# Ineffective Theorems and Higher-order Games

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(*based on joint work with M. Escardó and T. Powell*)

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# Plan

- Functional interpretation
- Game interpretation of functional interpretation
- Examples
  - Logic: *Drinker's paradox*
  - Arithmetic: *Infinite PHP*
  - Analysis: *Countable choice*

# Drinker's Paradox

$$\exists x^X(\phi(x) \to \forall y^X \phi(y))$$

$$\boxed{\exists \varepsilon^{(X \to X) \to X} \forall p^{X \to X} (\phi(\varepsilon p) \to \phi(p(\varepsilon p)))}$$

$$\varepsilon(p) = \begin{cases} a & \text{if } \phi(pa) \\ pa & \text{if } \neg\phi(pa) \end{cases}$$

# Drinker's Paradox

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# Infinite PHP

$$\forall n \forall c^{\mathbb{N}\to n} \exists i < n \forall j \exists k (k \geq j \wedge c(k) = i)$$

$$\boxed{\forall n \forall c^{\mathbb{N}\to n} \forall \varepsilon \exists i < n \exists p ( p(\varepsilon_i p) \geq \varepsilon_i p \ \ \wedge c(p(\varepsilon_i p)) = i )}$$

**we can produce witness $i$ and $p$ for this**

**but can we make sense of it?!?!**

# The Interpretation

$$\exists x^{X} \forall r^{R} \phi(x,r) \quad \Rightarrow \quad \exists \varepsilon^{(X \to R) \to X} \forall p^{X \to R} \phi(\varepsilon p, p(\varepsilon p))$$

# The Interpretation

## (of the interpretation)

game outcome

game continuation

$$\exists x^{X} \forall r^{R} \phi(x, r) \quad \Rightarrow \quad \exists \varepsilon^{(X \to R) \to X} \forall p^{X \to R} \phi(\varepsilon p, p(\varepsilon p))$$

player!

game move

good outcome given move

# Games

$x \quad y \quad z$

a game play

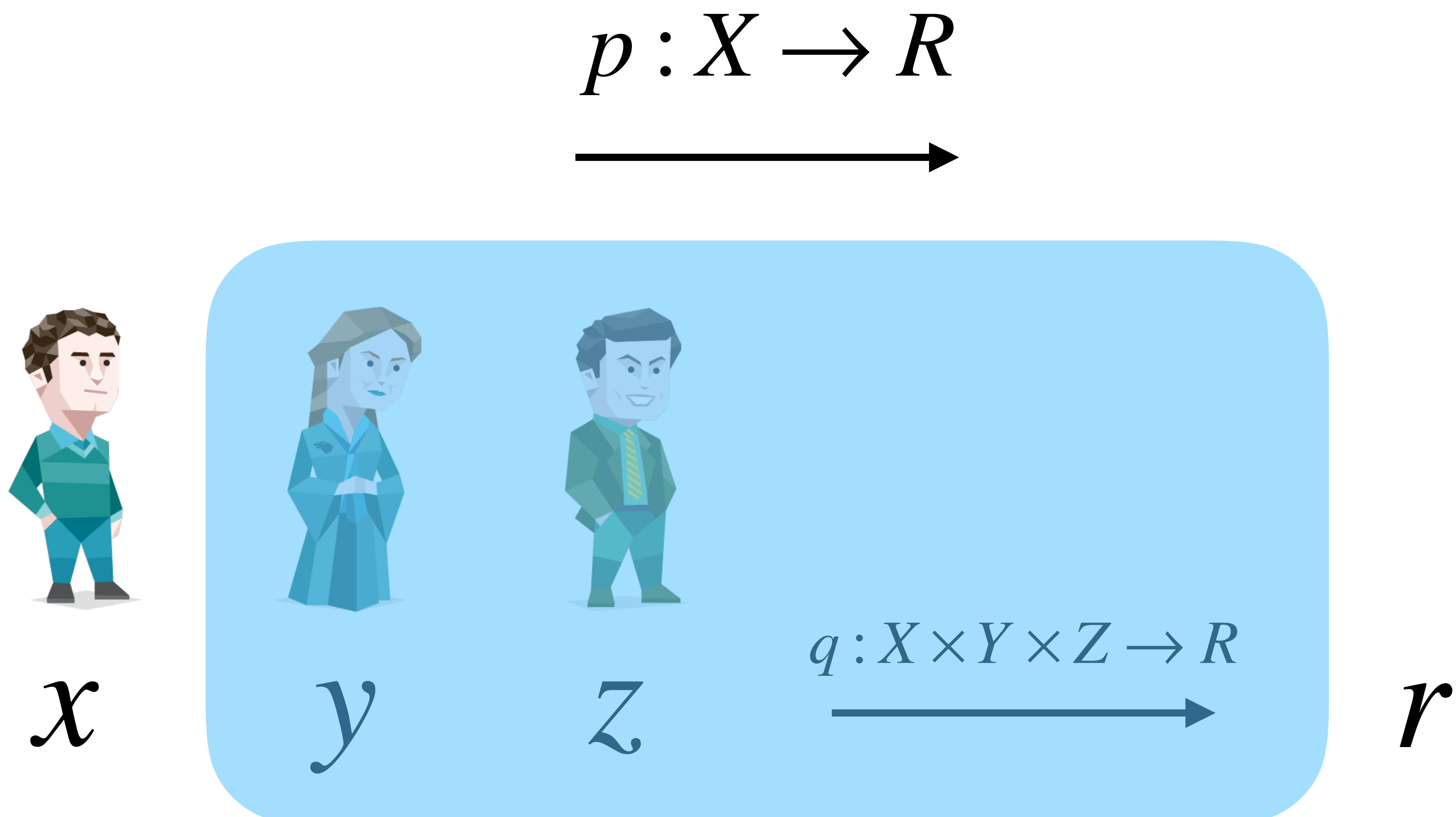
outcome function

$$q : X \times Y \times Z \to R$$

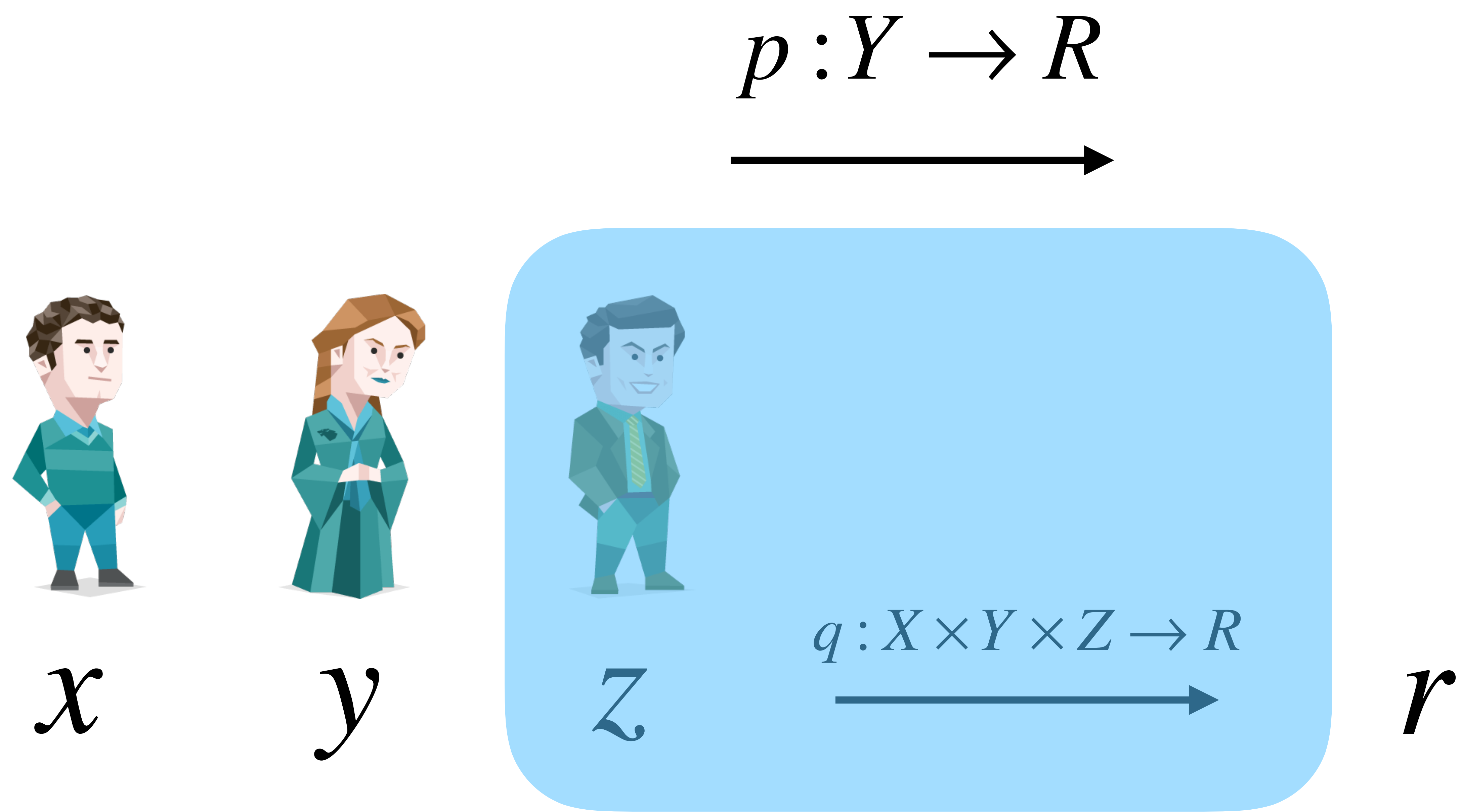
$r$

a game outcome

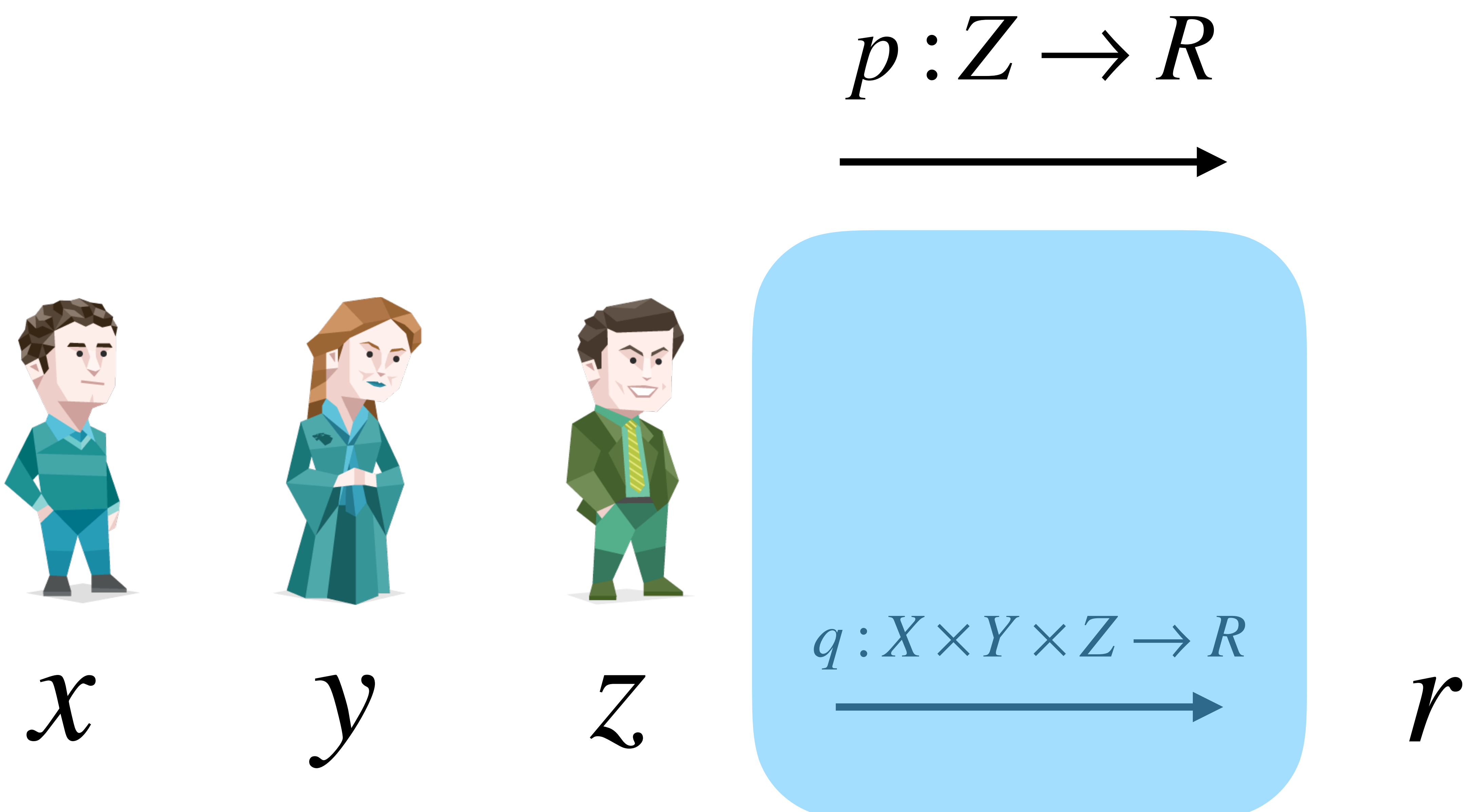
# Game Continuation



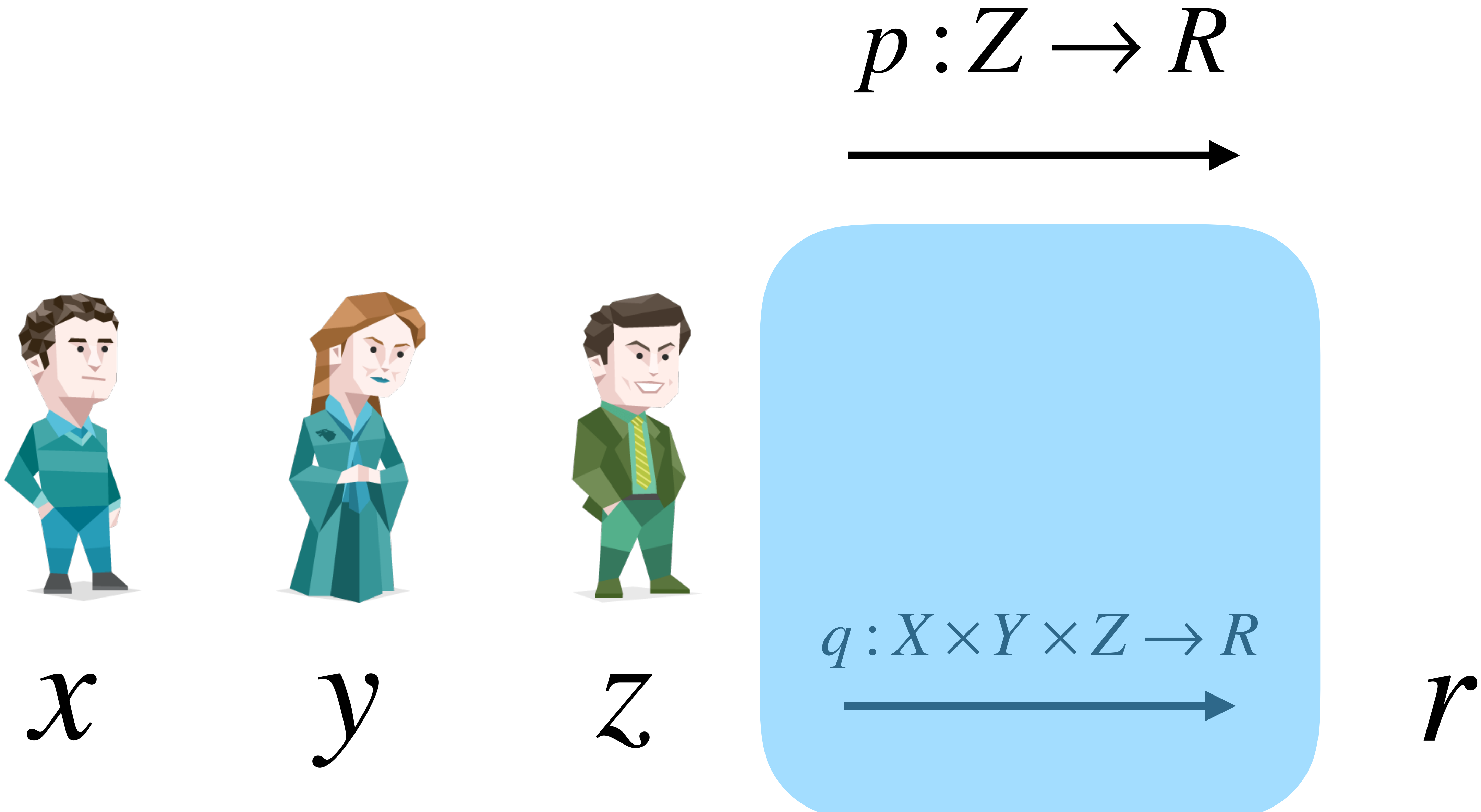
# Game Continuation



# Game Continuation



# Game Continuation



# Player

$$(X \to R) \to X$$

$$(Y \to R) \to Y$$

$$(Z \to R) \to Z$$

# Game

$$\varepsilon_X : \quad (X \to R) \to X$$

$$\varepsilon_Y : \quad (Y \to R) \to Y$$

$$\varepsilon_Z : \quad (Z \to R) \to Z$$

$$q : \quad X \times Y \times Z \to R$$

# Optimal Strategies

A strategy $\{\xi_i : \Pi_{j<i} X_j \to X_i\}_{i \leq n}$, is optimal if

$$\xi_i(x_1,\ldots,x_{i-1}) = \varepsilon_i(p_i)$$

where $p_i(y) = q(x_1,\ldots,x_{i-1},y,\xi_{i+1}(x_1,\ldots,x_{i-1},y),\ldots)$

Optimal strategies always exist and can be calculated as

$$\xi_i(x_1,\ldots,x_{i-1}) = \text{first}((\otimes_{j=i}^{n} \varepsilon_j)(q_{x_1,\ldots,x_{i-1}}))$$

where $\otimes_{j=i}^{n} \varepsilon_j$ is the product of selection functions

# Central Bank Paradox

$$\exists \varepsilon^{(X\to X)\to X} \forall p^{X\to X} (\phi(\varepsilon p) \to \phi(p(\varepsilon p)))$$

$X =$ rate of inflation

player = central bank

$\phi(x) =$ inflation target

continuation $= X \to X$

predicted inflation

real inflation

$$\varepsilon(p) = \left\{ \begin{array}{ll} 0\% & \text{if } \phi(p(0\%)) \\ p(0\%) & \text{otherwise} \end{array} \right\}$$

# Infinite PHP

$$\forall n \forall c^{\mathbb{N}\to n} \forall \varepsilon \exists i < n \exists p \left( p(\varepsilon_i p) \geq \varepsilon_i p \;\wedge c(p(\varepsilon_i p)) = i \right)$$

*n* players

one of the players

a game continuation

$r \geq x_i$

$c(r) = i$

$x_i = \varepsilon_i p$ (move of the $i$-th player)

$r = p(x_i)$ (game outcome)

# Countable Choice

$$\forall n^{\mathbb{N}} \exists x^{X} \forall y^{R} A_n(x, y) \to \exists \alpha \forall n \forall y A_n(\alpha n, y)$$

$$\forall \varepsilon, q, \omega (\forall n \forall p A_n(\varepsilon_n p, p(\varepsilon_n p)) \to \exists \alpha A_{\omega\alpha}(\alpha(\omega\alpha), q\alpha))$$

unbounded game

assumption about the players

play in the game

round

game outcome

# Benefits

- Functional interpretation of ineffective theorems are themselves mathematically interesting
(theorems about higher-order games)

- Game-theoretic intuition for the extraction process

- Extracted (higher-order) programs make more sense

- Bar recursion = calculation of optimal strategies
(generalisation of sub-game perfect equilibrium)

# References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