# From intuitionistic affine logic to classical logic, and back

Workshop on Proof Theory and its Applications
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(*based on joint work with Rob Arthan*)

# Minimal Affine Logic

$$\frac{}{\Delta, A \vdash A}$$

identy axiom (allows weakening)

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}$$

# Minimal Affine Logic

**conjunction introduction and elimination**

$$\frac{}{\Delta, A \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \rightarrow B}{\Gamma, \Delta \vdash B}$$

# Minimal Affine Logic

$$\frac{}{\Delta, A \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

**implication introduction and elimination**

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \rightarrow B}{\Gamma, \Delta \vdash B}$$

# Intuitionistic Affine Logic

$$\frac{}{\Delta, A \vdash A}$$

$$\frac{\Delta \vdash \bot}{\Delta \vdash A}$$

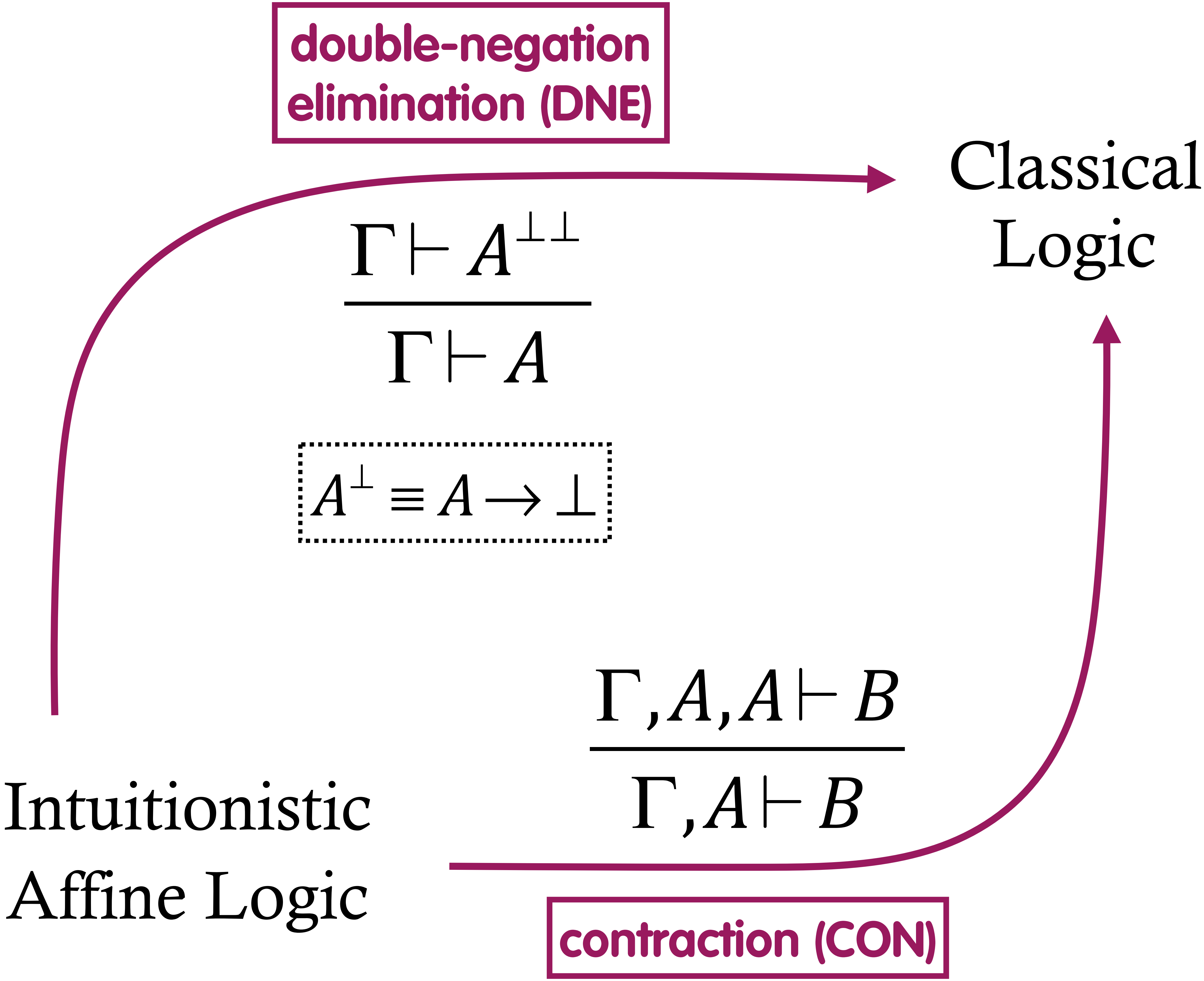
**ex falso quodlibet**

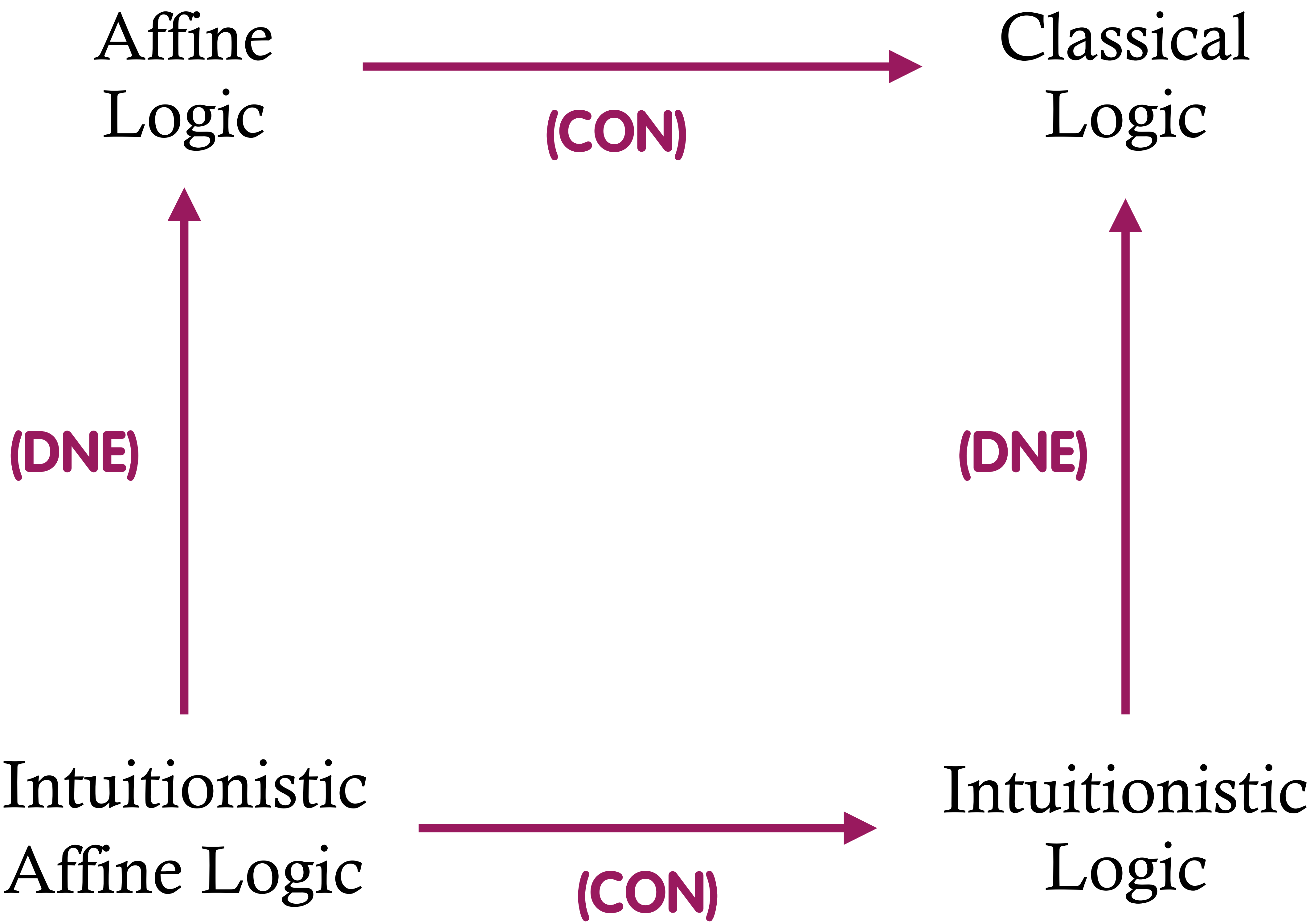
$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\frac{\Gamma \vdash A \otimes B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}$$


double-negation elimination (DNE)
Classical Logic
$\Gamma \vdash A^{\perp\perp}$
$\Gamma \vdash A$
$A^{\perp} \equiv A \rightarrow \perp$
$\Gamma, A, A \vdash B$
$\Gamma, A \vdash B$
Intuitionistic Affine Logic
contraction (CON)



Affine Logic
Classical Logic
(CON)
(DNE)
(DNE)
Intuitionistic Affine Logic
Intuitionistic Logic
(CON)

## (CON)

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

## (DNE)

$$\frac{\Gamma \vdash A^{\perp\perp}}{\Gamma \vdash A}$$

↓

## Divisibility (DIV)

$$\frac{\Gamma, A \vee B, B \rightarrow A \vdash C}{\Gamma, A \vdash C}$$

$$A \vee B \equiv (A \rightarrow B) \rightarrow B$$

↓

## Pre-linearity (LIN)

$$\frac{\Gamma, A \rightarrow B \vdash C \quad \Delta, B \rightarrow A \vdash C}{\Gamma, \Delta, C^{\perp} \vdash C}$$

# (CON) implies (DIV)

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{A \to B \vdash A \to B}}{A, A \to B \vdash B}}{A \vdash A \vee B}$$

weakening

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \to A} \qquad \Gamma, B \to A, A \vee B \vdash C}{\Gamma, A, A \vee B \vdash C}$$

$$\dfrac{A \vdash A \vee B \qquad \Gamma, A, A \vee B \vdash C}{\dfrac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}}$$

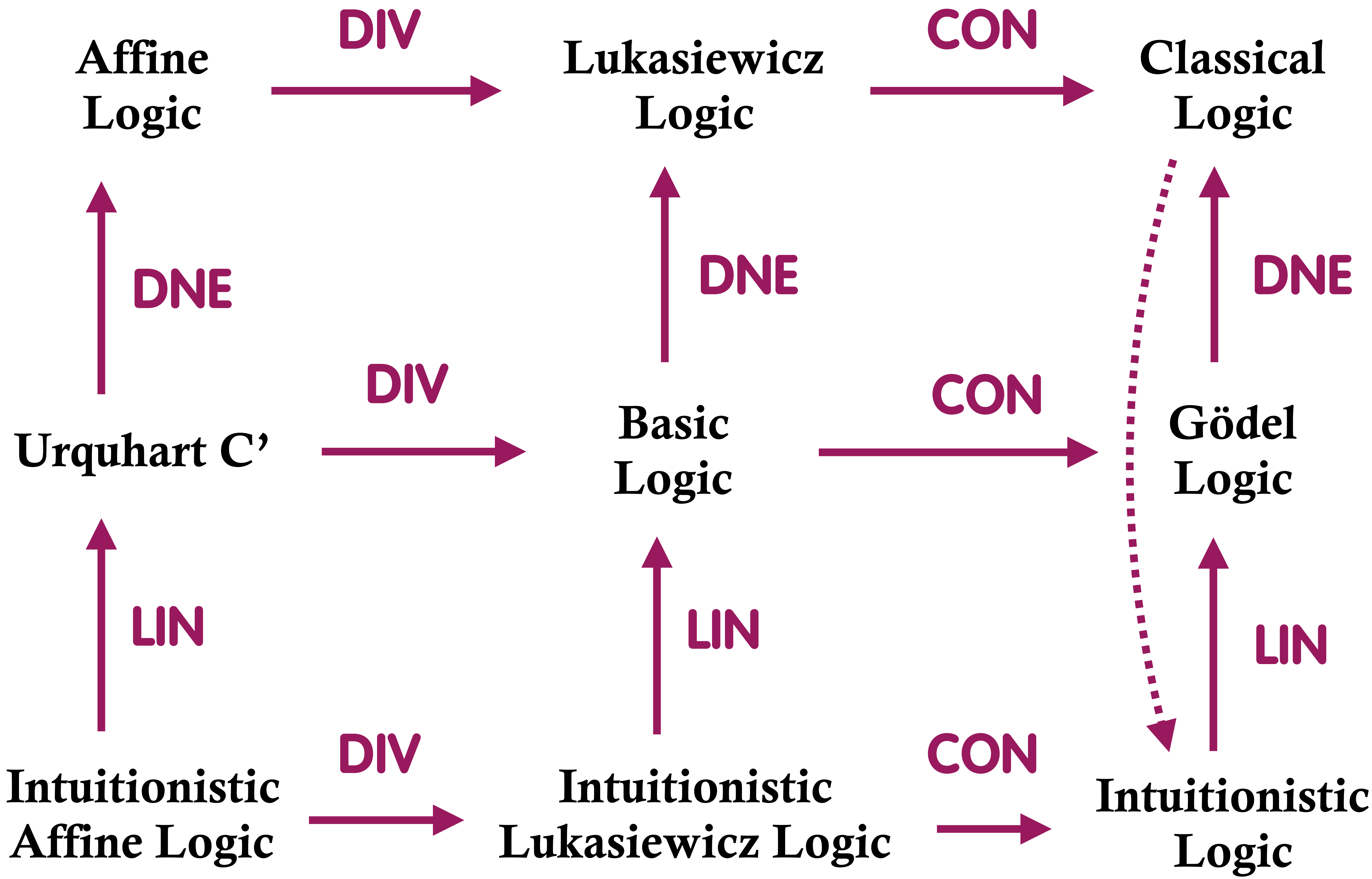
CON

$$\boxed{\dfrac{\Gamma \vdash A^{\perp\perp}}{\Gamma \vdash A}} \quad \Longrightarrow \quad \boxed{\dfrac{\Gamma, A \to B \vdash C \qquad \Delta, B \to A \vdash C}{\Gamma, \Delta, C^{\perp} \vdash C}}$$

**(DNE)** **(LIN)**

**weakening** **EFQ** **DNE**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B]_{\alpha}}{A \to B} \qquad (A \to B) \to C}{C} \qquad [C^{\perp}]_{\beta}}{\perp}}{A}}{B \to A}\,\alpha \qquad (B \to A) \to C}{C} \qquad C^{\perp}}{\perp}}{C^{\perp\perp}}\,\beta}{C}$$




R. Arthan and P. Oliva, **Negative translations for affine and Lukasiewicz logic**, Submitted for publication

**Classical Logic** ⇢ **Intuitionistic Logic**

$$P \equiv P^{\perp\perp}$$
$$(A \to B)^K \equiv (A^K \to B^K)^{\perp\perp}$$
$$(A \otimes B)^K \equiv (A^K \otimes B^K)^{\perp\perp}$$

$$A^{Gliv} \equiv A^{\perp\perp}$$

$$P^{Gen} \equiv P^{\perp\perp}$$
$$(A \to B)^{Gen} \equiv A^{Gen} \to B^{Gen}$$
$$(A \otimes B)^{Gen} \equiv A^{Gen} \otimes B^{Gen}$$

$$P \equiv P$$
$$(A \to B)^* \equiv A^* \to (B^*)^{\perp\perp}$$
$$(A \otimes B)^* \equiv A^* \otimes B^*$$
$$A^{Gödel} \equiv (A^*)^{\perp\perp}$$

If CL $\vdash A$ then IL $\vdash A^\dagger$, where $\dagger \in \{K, Gliv, Gen, Gödel\}$

**Affine Logic** → **Intuitionistic Affine Logic**

$$P \equiv P^{\perp\perp}$$
$$(A \to B)^K \equiv (A^K \to B^K)^{\perp\perp}$$
$$(A \otimes B)^K \equiv (A^K \otimes B^K)^{\perp\perp}$$

$$A^{Gliv} \equiv A^{\perp\perp}$$

$$P^{Gen} \equiv P^{\perp\perp}$$
$$(A \to B)^{Gen} \equiv A^{Gen} \to B^{Gen}$$
$$(A \otimes B)^{Gen} \equiv A^{Gen} \otimes B^{Gen}$$

$$P \equiv P$$
$$(A \to B)^* \equiv A^* \to (B^*)^{\perp\perp}$$
$$(A \otimes B)^* \equiv A^* \otimes B^*$$
$$A^{Gödel} \equiv (A^*)^{\perp\perp}$$

If $\mathrm{AL} \vdash A$ then $\mathrm{AL}_i \vdash A^{\dagger}$, where $\dagger \in \{K, Gödel\}$

## Lukasiewicz Logic → Intuitionistic Lukasiewicz Logic

$$P \equiv P^{\perp\perp}$$
$$(A \to B)^K \equiv (A^K \to B^K)^{\perp\perp}$$
$$(A \otimes B)^K \equiv (A^K \otimes B^K)^{\perp\perp}$$

$$P^{Gen} \equiv P^{\perp\perp}$$
$$(A \to B)^{Gen} \equiv A^{Gen} \to B^{Gen}$$
$$(A \otimes B)^{Gen} \equiv A^{Gen} \otimes B^{Gen}$$

$$A^{Gliv} \equiv A^{\perp\perp}$$

$$P \equiv P$$
$$(A \to B)^* \equiv A^* \to (B^*)^{\perp\perp}$$
$$(A \otimes B)^* \equiv A^* \otimes B^*$$
$$A^{Gödel} \equiv (A^*)^{\perp\perp}$$

If $\mathrm{LL} \vdash A$ then $\mathrm{LL}_i \vdash A^\dagger$, where $\dagger \in \{K, Gliv, Gen, Gödel\}$

Glivenko and Gentzen translations rely on the following homomorphism properties:

$$(A \otimes B)^{\perp\perp} \Leftrightarrow A^{\perp\perp} \otimes B^{\perp\perp}$$

$$(A \rightarrow B)^{\perp\perp} \Leftrightarrow A^{\perp\perp} \rightarrow B^{\perp\perp}$$

These are valid in intuitionistic logic, but proof relies on contraction, e.g…

$$\boxed{(A\otimes B)^{\perp\perp} \vdash A^{\perp\perp} \otimes B^{\perp\perp}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\otimes B \vdash A \qquad A^{\perp} \vdash A^{\perp}}{A\otimes B, A^{\perp} \vdash \bot}}{A^{\perp} \vdash (A\otimes B)^{\perp}} \qquad \dfrac{}{(A\otimes B)^{\perp\perp} \vdash (A\otimes B)^{\perp\perp}}}{A^{\perp}, (A\otimes B)^{\perp\perp} \vdash \bot}}{(A\otimes B)^{\perp\perp} \vdash A^{\perp\perp}} \qquad \dfrac{\cdots}{(A\otimes B)^{\perp\perp} \vdash B^{\perp\perp}}}{(A\otimes B)^{\perp\perp}, (A\otimes B)^{\perp\perp} \vdash A^{\perp\perp} \otimes B^{\perp}}$$

$$(A \otimes B)^{\perp\perp} \Leftrightarrow A^{\perp\perp} \otimes B^{\perp\perp}$$

$$(A \to B)^{\perp\perp} \Leftrightarrow A^{\perp\perp} \to B^{\perp\perp}$$

*Question*: Are these derivable using **DIV**?

*Answer*: Yes!

*Semantics*: Intuitionistic Lukasiewicz logic has an algebraic semantics. It corresponds to hoops (*partially ordered commutative residuated integral monoids with divisibility*)

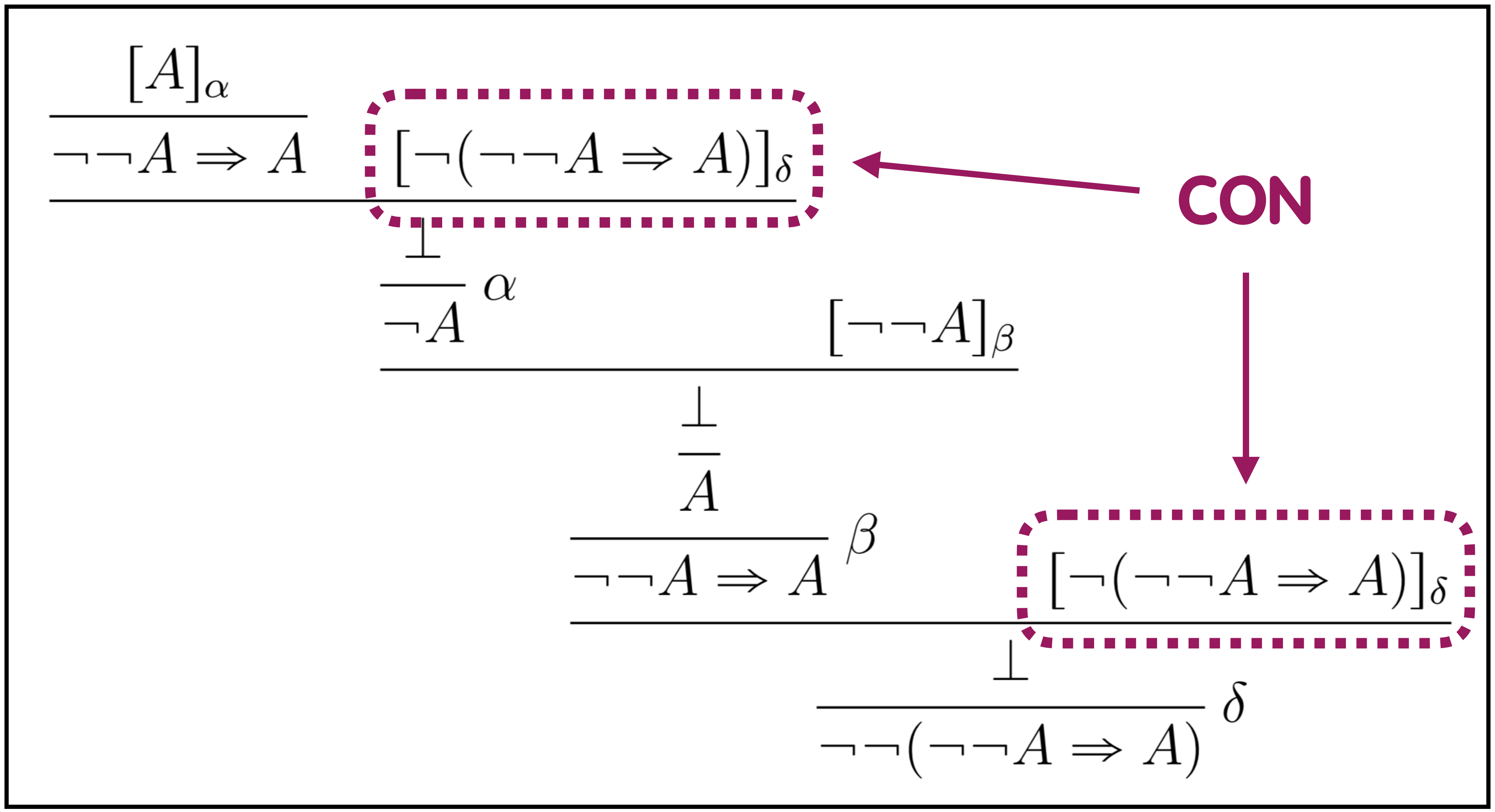
*Method*: Use Prover9 to search for derivation of corresponding equation in the theory of hoops

*Result*: Proofs found, but long. A lot of work to make them human readable.


R. Arthan and P. Oliva, **Studying algebraic structures using prover9 and mace4**
In: Proof Technology in Mathematics Research and Teaching, 2018

# Demo:Using prover9 and mace4

Consider the IL derivation of $\neg\neg(\neg\neg A \Rightarrow A)$




Can we replace **CON** by **DIV**?

# Some Questions

- Shorter/simpler proofs of homomorphism properties?
- Finer negative translation for **LIN**?
- **CON**-elimination procedure?
- Cut-free system for Intuitionistic Lukasiewicz Logic?
- Curry-Howard for ILL with normalisation?


R. Arthan and P. Oliva, **A Curry-Howard correspondence for the minimal fragment Lukasiewicz logic**, report, 2017

R. Arthan and P. Oliva, **On affine logic and Lukasiewicz logic**, Available on arXiv (https://arxiv.org/abs/1404.0570)