On “Approximate” Variants of Functional Interpretations

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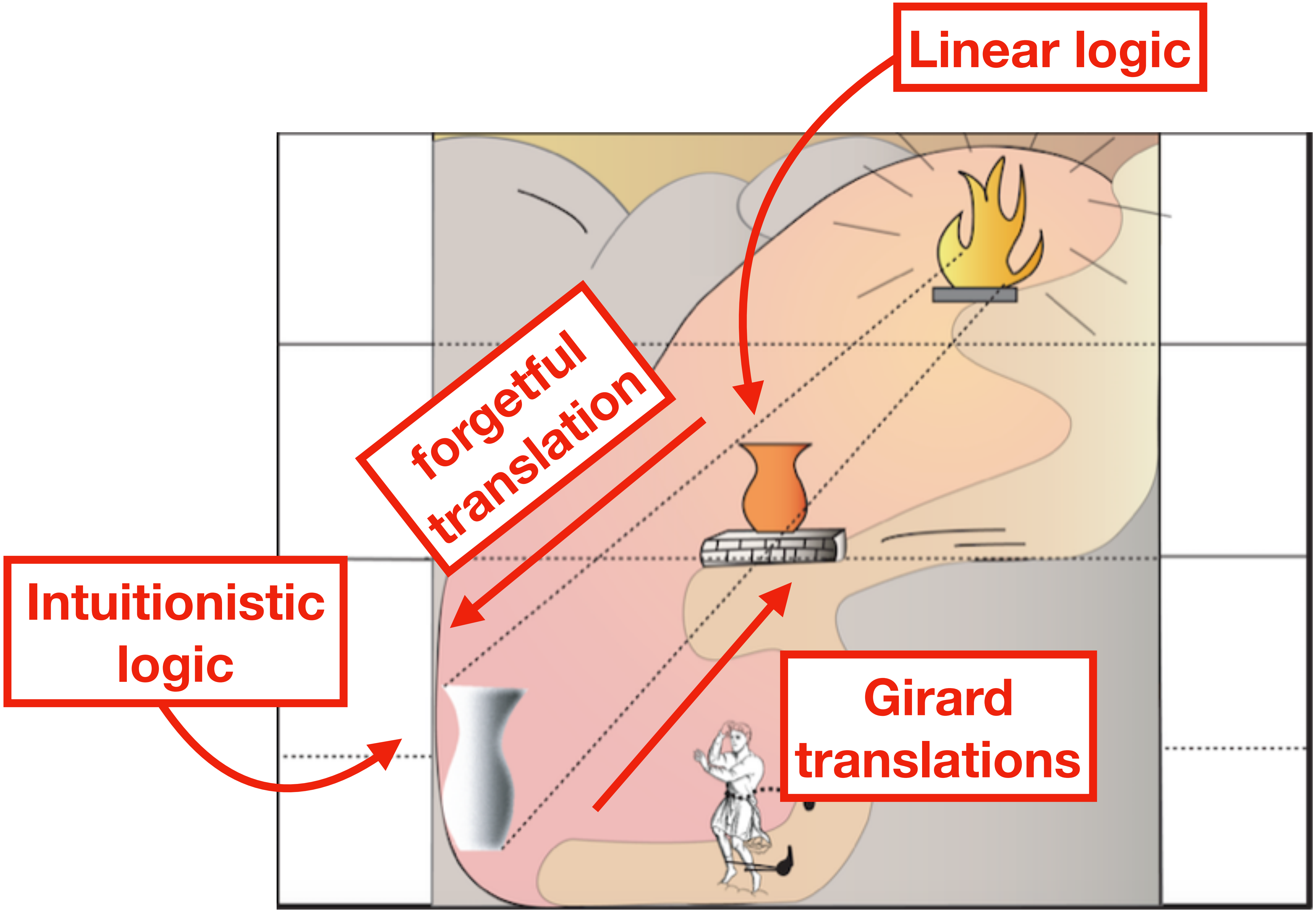
Proof, Computation, Complexity

Djursholm, 19 July 2019

“*Far better an approximate answer to the right question than the exact answer to the wrong question.*”

- John Tukey


Linear logic
forgetful translation
Intuitionistic logic
Girard translations

Plan

1. Functional interpretations
2. Precise vs approximate
3. Analysis of precise interpretations
4. Analysis of approximate interpretations

1.
Functional Interpretations

Functional Interpretations

$$\text{formula } A \xrightarrow{\;I_{\mathcal{F}}\;} \begin{array}{c}\text{functional}\\ \text{specification}\\ I_{\mathcal{F}}(A)\end{array}$$

$$\text{proof } \pi_A \text{ of } A \xrightarrow{\;I_{\mathcal{P}}\;} \begin{array}{c}\text{program}\\ I_{\mathcal{P}}(\pi_A) \in I_{\mathcal{F}}(A)\end{array}$$

$$I_{\mathcal{F}}(A) = \langle x^X ; y^Y ; \underbrace{|A| \subseteq X \times Y}_{|A|^x_y} \rangle$$

argument

counter-argument

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

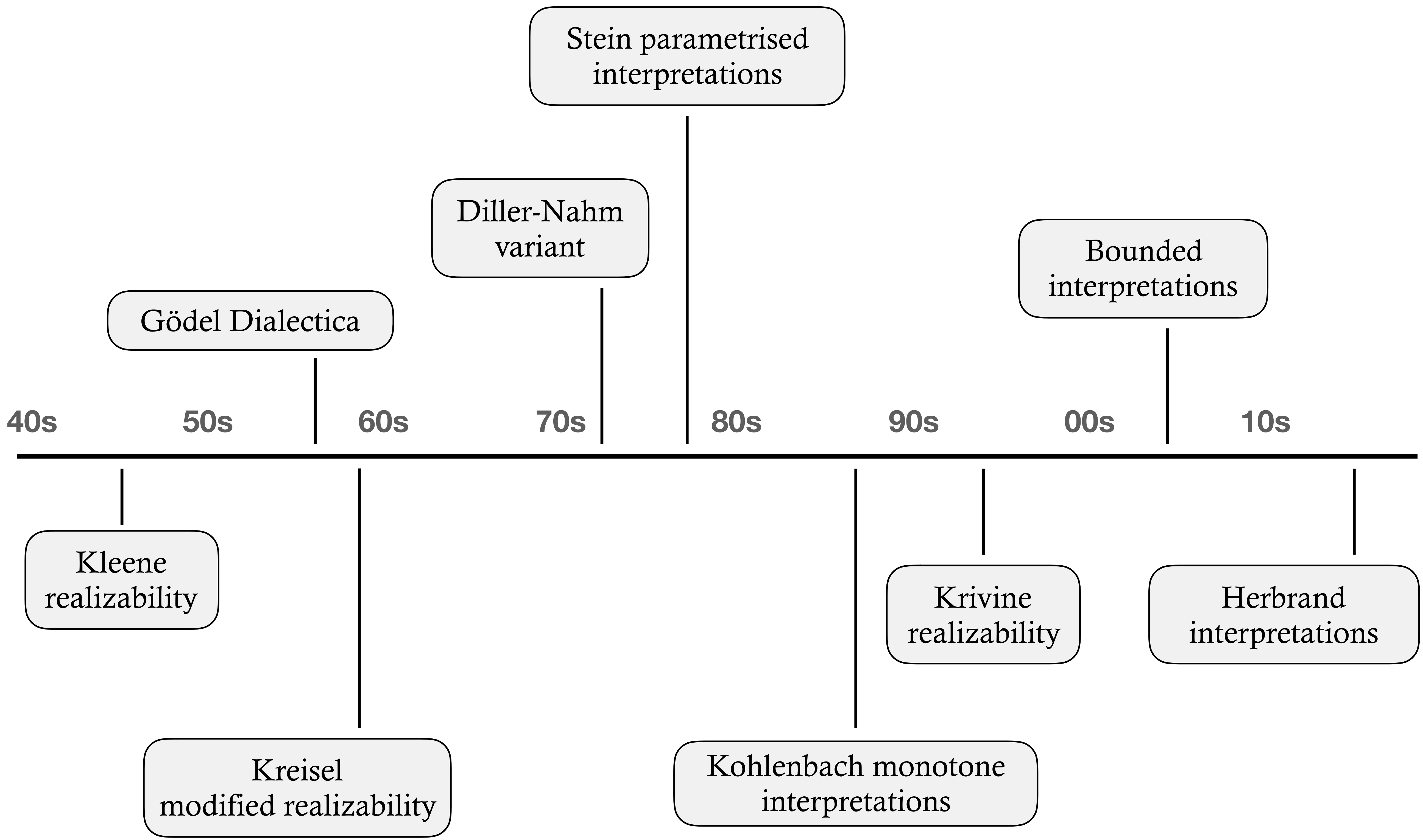
$$|A \vee B|^{x,v,b}_{y,w} \equiv (b=0 \wedge |A|^x_y) \vee (b \neq 0 \wedge |B|^v_w)$$

$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \rightarrow |B|^{f(x)}_w$$

$$|\forall z A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z A(z)|^{x,z}_y \equiv |A(z)|^x_y$$

1. If $\mathrm{HA}^\omega \vdash A$ then $\mathrm{T} \vdash \forall y |A|^t_y$, for t in system T
2. $\mathrm{HA}^\omega + \mathrm{IP}_\forall + \mathrm{MP} + \mathrm{AC} \vdash A \leftrightarrow \exists x \forall y |A|^x_y$


Stein parametrised interpretations
Diller-Nahm variant
Bounded interpretations
Gödel Dialectica
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Krivine realizability
Herbrand interpretations
Kreisel modified realizability
Kohlenbach monotone interpretations

2.
Precise Interpretations
vs
Approximate Interpretations

"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise

The number is 17

The function is $\lambda x \,.\, \frac{x}{2}$

Approximate

It's an odd number

It's less than 50

It's either 11 or 17

The function is bounded by $\lambda x \,.\, x^2$

It's either $\lambda x \,.\, \frac{x}{2}$ or $\lambda x \,.\, x^2$

"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise

$$f(x) = \begin{cases} 0 & \text{if } \exists u. T(x, x, u) \\ 1 & \text{if } \forall u. \neg T(x, x, u) \end{cases}$$

precise

but

non-computable

Approximate

$$g(x) = \{0, 1\}$$

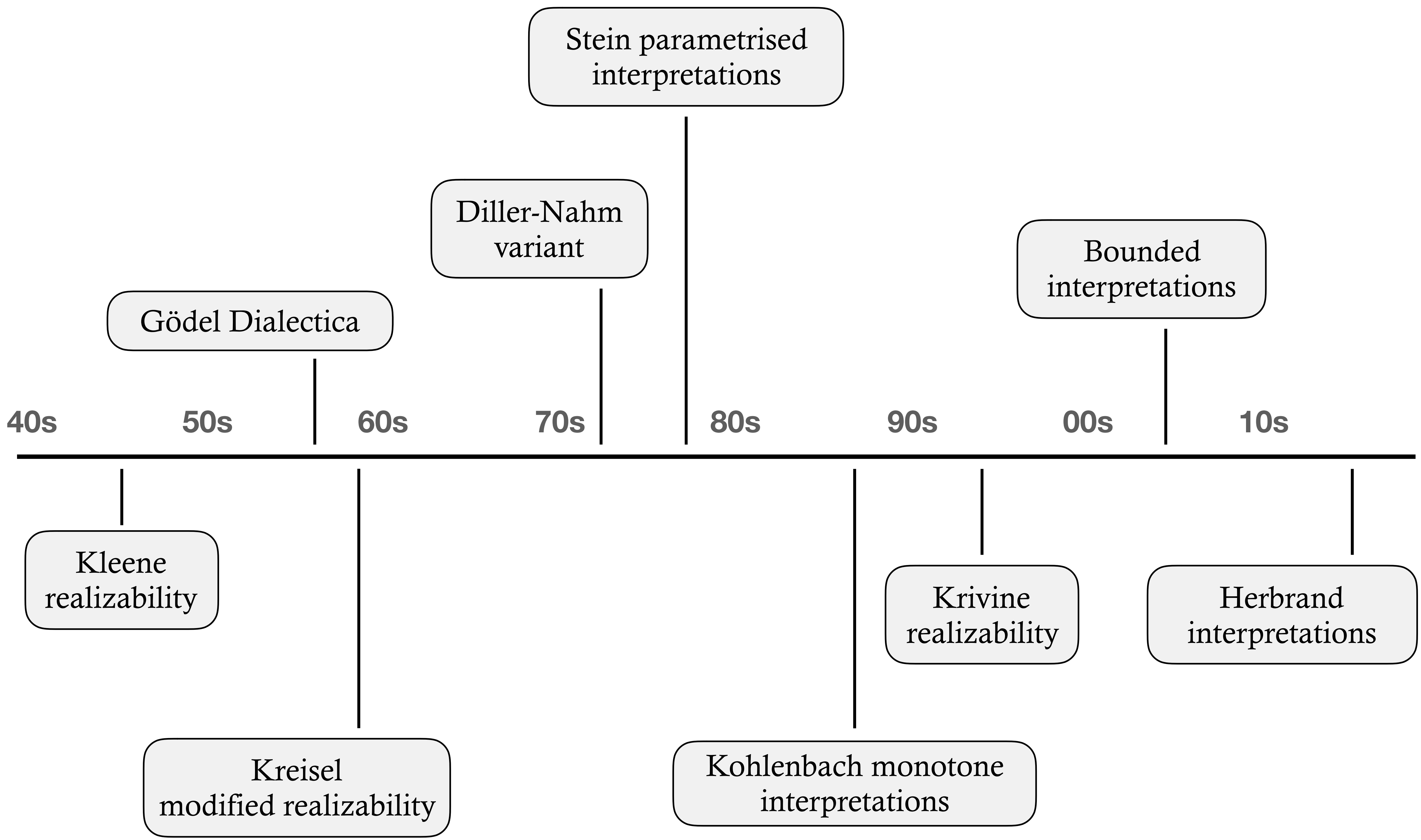
approximate

but

computable


precise
inputs
non-computable
inputs
proof/
program
precise
outputs
non-computable
outputs


approximate
inputs
computable
inputs
proof/
program
approximate
outputs
computable
outputs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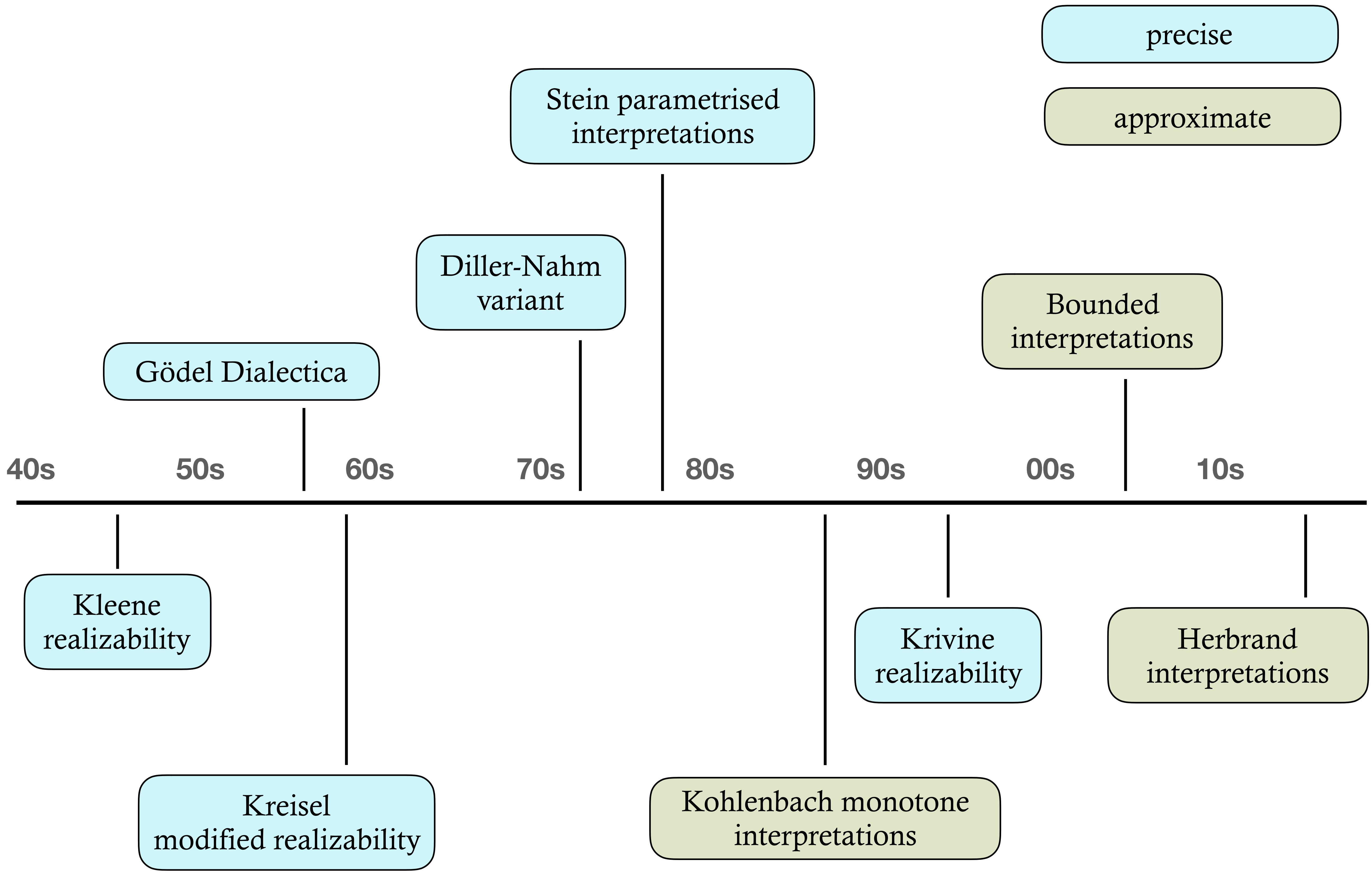
precise
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Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

"Herbrand" interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

What do they have in common?

In which way are they different?

Any others waiting to be discovered?

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

"Herbrand" interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

Result 1: Relational presentation of realizability

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

Result 2: Only differ in the treatment of contraction (!A)

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

Result 3: Multiple exponentials = combined interpretations

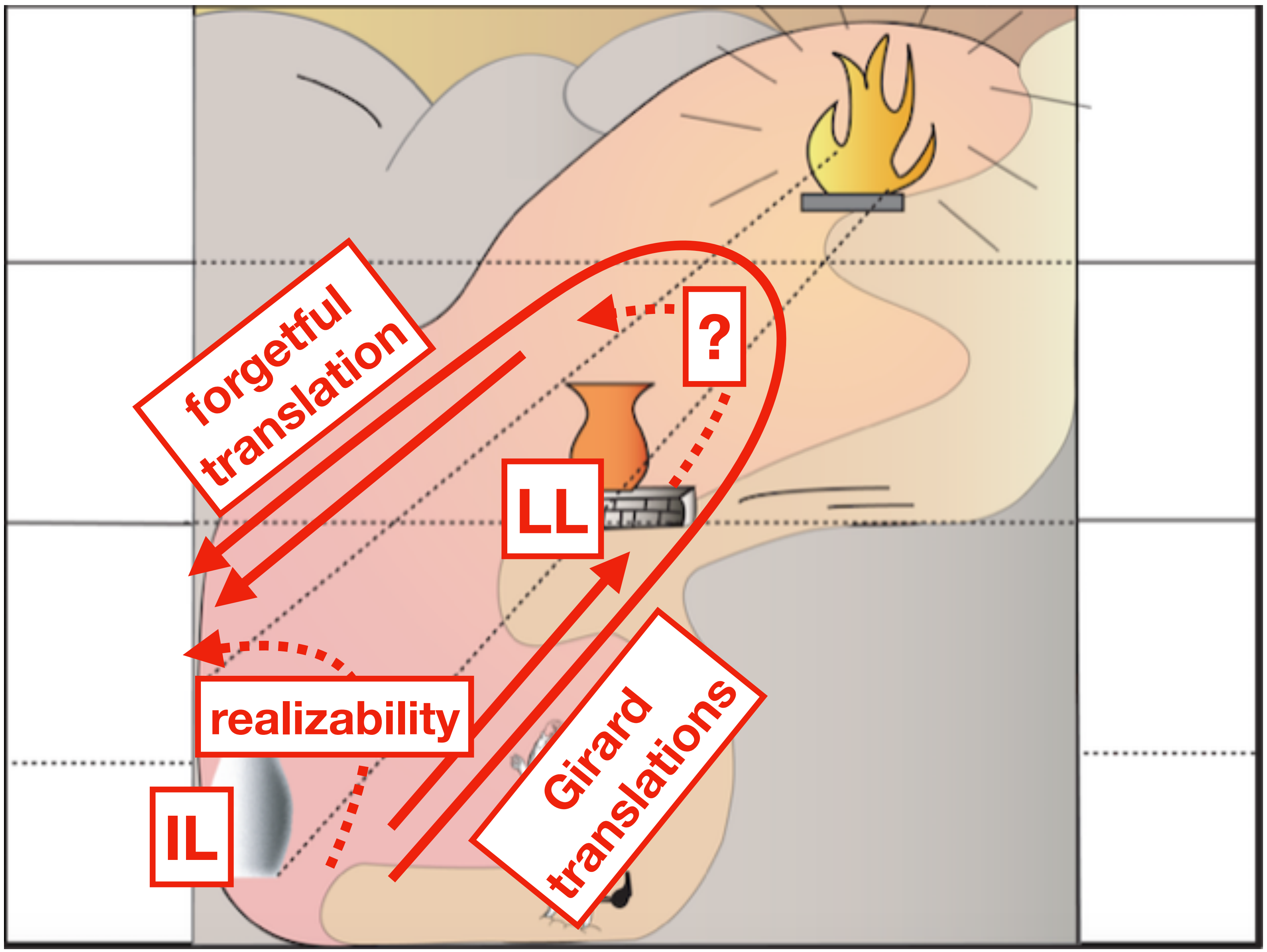
M.D. Hernest and P. Oliva, **Hybrid functional interpretations**, CiE, 2008

3.
Unifying Precise Interpretations through Linear Logic

call-by-name translation		
$(A \wedge B)^*$	$\equiv$	$A^* \,\&\, B^*$
$(A \vee B)^*$	$\equiv$	$!A^* \oplus !B^*$
$(A \to B)^*$	$\equiv$	$!A^* \multimap B^*$
$(\forall z A)^*$	$\equiv$	$\forall z A^*$
$(\exists z A)^*$	$\equiv$	$\exists z !A^*$

call-by-value translation		
$(A \wedge B)^\circ$	$\equiv$	$A^\circ \otimes B^\circ$
$(A \vee B)^\circ$	$\equiv$	$A^\circ \oplus B^\circ$
$(A \to B)^\circ$	$\equiv$	$!(A^\circ \multimap B^\circ)$
$(\forall z A)^\circ$	$\equiv$	$!\forall z A^\circ$
$(\exists z A)^\circ$	$\equiv$	$\exists z A^\circ$

$$\mathrm{IL} \vdash A \quad \Longrightarrow \quad \begin{array}{l} \mathrm{LL} \vdash A^\circ \\ \mathrm{LL} \vdash A^* \end{array}$$


forgetful translation
?
LL
realizability
IL
Girard translations

Interpretation of Linear Logic

$$|A \otimes B|^{x,v}_{y,w} \equiv |A|^x_y \otimes |B|^v_w$$

$$|A \oplus B|^{x,v,b}_{y,w} \equiv (b{=}0 \,\&\, |A|^x_y) \oplus (b \neq 0 \,\&\, |B|^v_w)$$

$$|A \,\&\, B|^{x,v}_{y,w,b} \equiv (b{=}0 \,\&\, |A|^x_y) \oplus (b \neq 0 \,\&\, |B|^v_w)$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \multimap |B|^{f(x)}_w$$

$$|\forall z\, A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z\, A(z)|^{x,z}_y \equiv |A(z)|^x_y$$

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "\text{modified realizability}"] \arrow[d, "(\cdot)^*"] & x \;\mathrm{mr}\; A \arrow[d, "(\cdot)^*"] \\
A^* \arrow[r, "|!A|^x \equiv !\forall y \, |A|^x_y"'] & (x \;\mathrm{mr}\; A)^* \Leftrightarrow \forall y \, |A^*|^x_y
\end{tikzcd}
$$

interpretations (only) differ in treatment of $!A$

!A	Trans.	Interpretation
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Kreisel modified realizability
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$\lvert !A \rvert^x_a \equiv ! \lvert A \rvert^x_a$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

4.
Unifying Approximate Interpretations through Linear Logic

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

"Herbrand" interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

How do these fit (if at all) into the "unification"?

Kohlenbach monotone interpretations

formula A $\xrightarrow{I_{\mathcal{F}}}$ *functional specification* $I_{\mathcal{F}}(A)$

proof π_A *of A* $\xrightarrow{I^*_{\mathcal{P}}}$ ***majorant*** $I^*_{\mathcal{P}}(\pi_A) \geq^* I_{\mathcal{F}}(A)$

$Dial_{\mathcal{F}} + Dial^*_{\mathcal{P}}$	$=$	monotone Dialectica
$Real_{\mathcal{F}} + Real^*_{\mathcal{P}}$	$=$	monotone modified realizability

Bounded functional interpretation

Fernando Ferreira[a,*], Paulo Oliva[b]

Annals of Pure and Applied Logic 135 (2005) 73–112

$$
\begin{aligned}
n \leq^* m &\equiv n \leq m \\
f \leq^* g &\equiv \forall a \forall x \leq^* a (fx \leq^* ga \wedge gx \leq^* ga)
\end{aligned}
$$

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^x_y \wedge |B|^v_w \\
|A \vee B|^{x,v}_{y,w} &\equiv (\forall y' \leq^* y \, |A|^x_{y'}) \vee (\forall w' \leq^* w \, |B|^v_{w'}) \\
|A \rightarrow B|^{f,g}_{x,w} &\equiv \forall y \leq^* g(x,w) \, |A|^x_y \rightarrow |B|^{f(x)}_w \\
|\forall z A(z)|^f_{y,c} &\equiv \forall z \leq^* c \, |A(z)|^{f(c)}_z \\
|\exists z A(z)|^{x,c}_y &\equiv \exists z \leq^* c \forall y' \leq^* y \, |A(z)|^x_{y'}
\end{aligned}
$$

A functional interpretation for nonstandard arithmetic

Benno van den Berg[a,*,1], Eyvind Briseid[b,2], Pavol Safarik[c,3]

Annals of Pure and Applied Logic 163 (2012) 1962–1994

$$f[x] :\equiv \bigcup_{f' \in f} f'(x)$$

$$|A \wedge B|^{x,v}_{y,w} \quad\equiv\quad |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \quad\equiv\quad |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \quad\equiv\quad \forall y \in g[x,w]\, |A|^{x}_{y} \to |B|^{f[x]}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \quad\equiv\quad |A(z)|^{f[z]}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \quad\equiv\quad \exists z' \in z\, \forall y' \in y\, |A(z')|^{x}_{y'}$$

Bounded functional interpretation

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v}_{y,w} &\equiv (\forall y' \leq^* y\, |A|^{x}_{y'}) \vee (\forall w' \leq^* w\, |B|^{v}_{w'}) \\
|A \to B|^{f,g}_{x,w} &\equiv \forall y \leq^* g(x,w)\, |A|^{x}_{y} \to |B|^{f(x)}_{w} \\
|\forall z A(z)|^{f}_{y,c} &\equiv \forall z \leq^* c\, |A(z)|^{f(c)}_{z} \\
|\exists z A(z)|^{x,c}_{y} &\equiv \exists z \leq^* c \forall y' \leq^* y\, |A(z)|^{x}_{y'}
\end{aligned}
$$

Herbrand functional interpretation

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \vee |B|^{v}_{w} \\
|A \to B|^{f,g}_{x,w} &\equiv \forall y \in g[x,w]\, |A|^{x}_{y} \to |B|^{f[x]}_{w} \\
|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} &\equiv |A(z)|^{f[z]}_{y} \\
|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} &\equiv \exists z' \in z \forall y' \in y\, |A(z')|^{x}_{y'}
\end{aligned}
$$

Result: Herbrand interpretations have alternative presentations that use usual functional application

Herbrand functional interpretation (alternative presentation)

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g(x,w) |A|^{x}_{y} \to |B|^{f(x)}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \equiv |A(z)|^{f(z)}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \equiv \exists z' \in z \forall y' \in y |A(z')|^{x}_{y'}$$

Consequence 1: Arguments are not always sets

Consequence 2: Full monotonicity property no longer holds

$$
\begin{array}{lcl}
(s^{\rho^*})^{(\mathrm{st}^\sigma(z))\uparrow} & :\equiv & s \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A}, \boldsymbol{b}^{\boldsymbol{\tau}^d_B})^{(A\wedge B)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}, (\boldsymbol{b}^{\boldsymbol{\tau}^d_B})^{B\uparrow} \\
(\boldsymbol{\phi}^{\boldsymbol{\tau}^d_A \to \boldsymbol{\tau}^d_B})^{(A\to B)\uparrow} & :\equiv & \{\lambda \boldsymbol{a}^{\boldsymbol{\tau}^u_A}.(\boldsymbol{\phi}(\boldsymbol{a}^{A\downarrow}))^{B\uparrow}\} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A}, c^{\sigma^*})^{(\exists z^{\mathrm{st}^\sigma} A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}, c \\
(\boldsymbol{\phi}^{\sigma^* \to \boldsymbol{\tau}^d_A})^{(\forall z^{\mathrm{st}^\sigma} A)\uparrow} & :\equiv & \{\lambda c^{\sigma^*}.(\boldsymbol{\phi}(c))^{A\uparrow}\} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\exists z^\sigma A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\forall z^\sigma A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}
\end{array}
$$

For any $\boldsymbol{a} \colon \boldsymbol{\tau}^d_A$ *we have* $(\boldsymbol{a}^{A\uparrow})^{A\downarrow} =_{\boldsymbol{\tau}^d_A} \boldsymbol{a}$

$$
\begin{array}{ll}
(i) & \boldsymbol{a} \ \mathrm{hr}' \ A \ \Rightarrow \ \boldsymbol{a}^{A\uparrow} \ \mathrm{hr} \ A \\
(ii) & \boldsymbol{a} \ \mathrm{hr} \ A \ \Rightarrow \ \boldsymbol{a}^{A\downarrow} \ \mathrm{hr}' \ A
\end{array}
$$

$$
\begin{array}{lcl}
(s^{\rho^*})^{(\mathrm{st}^\sigma(z))\downarrow} & :\equiv & s \\
(\boldsymbol{a}^{\boldsymbol{\tau}^u_A}, \boldsymbol{b}^{\boldsymbol{\tau}^u_B})^{(A\wedge B)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^u_A})^{A\downarrow}, (\boldsymbol{b}^{\boldsymbol{\tau}^u_B})^{B\downarrow} \\
(\boldsymbol{\phi}^{(\boldsymbol{\tau}^u_A \to \boldsymbol{\tau}^u_B)^*})^{(A\to B)\downarrow} & :\equiv & \lambda \boldsymbol{a}^{\boldsymbol{\tau}^d_A}.(\mathrm{U}^{\boldsymbol{\tau}^u_B}\{\boldsymbol{\phi}'(\boldsymbol{a}^{A\uparrow}) \ : \ \boldsymbol{\phi}' \in \boldsymbol{\phi}\})^{B\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^u_A}, c^{\sigma^*})^{(\exists z^{\mathrm{st}^\sigma} A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^u_A})^{A\downarrow}, c \\
(\boldsymbol{\phi}^{(\sigma^* \to \boldsymbol{\tau}^u_A)^*})^{(\forall z^{\mathrm{st}^\sigma} A)\downarrow} & :\equiv & \lambda c^{\sigma^*}.(\mathrm{U}^{\boldsymbol{\tau}^u_A}(\{(\boldsymbol{\phi}' c) \ : \ \boldsymbol{\phi}' \in \boldsymbol{\phi}\}))^{A\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\exists z^\sigma A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\forall z^\sigma A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\downarrow}
\end{array}
$$

P. Oliva, **Kreisel's modified realizability and recent variants**, to appear

Unifying Interpretation 2.0

$$
\begin{array}{lcl}
|P(x)|^{a} & \equiv & x \prec_P a \\
|A \otimes B|^{x,v}_{y,w} & \equiv & |A|^{x}_{y} \otimes |B|^{v}_{w} \\
|A \Diamond_z B|^{x,v}_{y,w} & \equiv & |A|^{x}_{y} \Diamond_z |B|^{v}_{w} \\
|A \multimap B|^{f,g}_{x,w} & \equiv & |A|^{x}_{g(x,w)} \multimap |B|^{f(x)}_{w} \\
|\forall z A(z)|^{x}_{y} & \equiv & \forall z |A(z)|^{x}_{y} \\
|\exists z A(z)|^{x}_{y} & \equiv & \exists z |A(z)|^{x}_{y} \\
|!A|^{x}_{a} & \equiv & \forall y \sqsubset a\, |A|^{x}_{y}
\end{array}
$$

one parameter for each predicate symbol P

one parameter to deal with contraction

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\forall x^\tau A \quad \equiv \quad \forall x(\tau(x) \to A)$$
$$\exists x^\tau A \quad \equiv \quad \exists x(\tau(x) \wedge A)$$

Precise

$$|\tau(x)|^a \quad \equiv \quad x = a$$

Approximate

$$|\tau(x)|^a \quad \equiv \quad x \leq^* a$$

$$|\tau(x)|^a \quad \equiv \quad x \in a$$

...

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\begin{array}{lcl} \forall x^{\tau,\mathrm{st}} A & \equiv & \forall x(\tau(x) \to \mathrm{st}(\mathrm{x}) \to A) \\ \exists x^{\tau,\mathrm{st}} A & \equiv & \exists x(\tau(x) \wedge \mathrm{st}(\mathrm{x}) \wedge A) \end{array}$$

Type predicate / Standardness predicate

$$\begin{array}{lcl} |\tau(x)| & \equiv & \tau(x) \\ |\mathrm{st}(x)|^a & \equiv & x \in a \end{array}$$

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

$\forall y \sqsubset a \, |A|^x_y$	$x \prec_P a$	**Interpretation**
$\forall y \, |A|^x_y$	$\tau(x) \wedge (x = a)$	Kreisel modified realizability
$\forall y \in a \, |A|^x_y$	$\tau(x) \wedge (x = a)$	Diller-Nahm interpretation
$|A|^x_a$	$\tau(x) \wedge (x = a)$	Gödel's Dialectica interpretation
$\forall y \, |A|^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded modified realizability
$\forall y \leq^* a \, |A|^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded functional interpretation
$\forall y \, |A|^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand modified realizability
$\forall y \in a \, |A|^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand functional interpretation

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P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

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