Unifying Functional Interpretations 2.0

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Oberseminar Mathematische Logik

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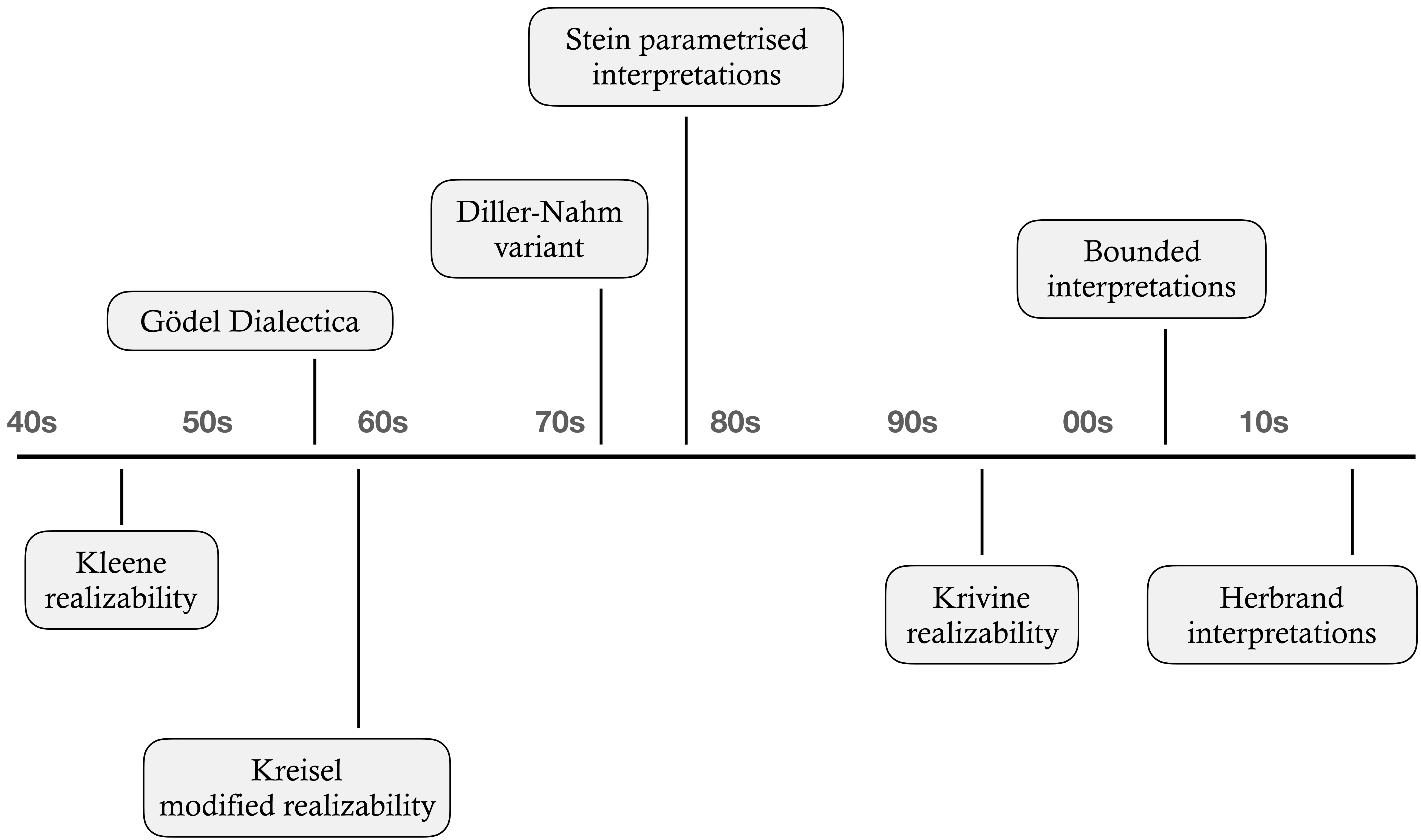

Stein parametrised interpretations
Diller-Nahm variant
Bounded interpretations
Gödel Dialectica
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Krivine realizability
Herbrand interpretations
Kreisel modified realizability

Plan

1. Functional interpretations
2. Unifying functional interpretations 1.0
3. Unifying functional interpretations 2.0

Functional Interpretations

Realizability and Functional Interpretations

formula A $\xrightarrow{\ I_{\mathcal{F}}\ }$ *functional specification* $I_{\mathcal{F}}(A)$

proof π_A *of A* $\xrightarrow{\ I_{\mathcal{P}}\ }$ *program* $I_{\mathcal{P}}(\pi_A) \in I_{\mathcal{F}}(A)$

$$\text{formula } A \xrightarrow{\ I_{\mathcal{F}}\ } \text{functional specification } I_{\mathcal{F}}(A)$$

$$A = \forall n \exists m (m = n^2)$$

$$I_{\mathcal{F}}(A) = \langle f;\ n;\ f(n) = n^2 \rangle$$

$$A = \sqrt{2} \notin \mathbb{Q}$$

$$I_{\mathcal{F}}(A) = \langle f;\ p,q;\ \left|\frac{p}{q} - \sqrt{2}\right| > f(p,q) \rangle$$

$$I_{\mathcal{F}}(A) = \langle f;\, n;\, \underbrace{f(n) = n^2} \rangle$$

$$|A|^f_n$$

argument

counter-argument

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt Gödel, Princeton

Dialectica, vol. 12, 1958

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

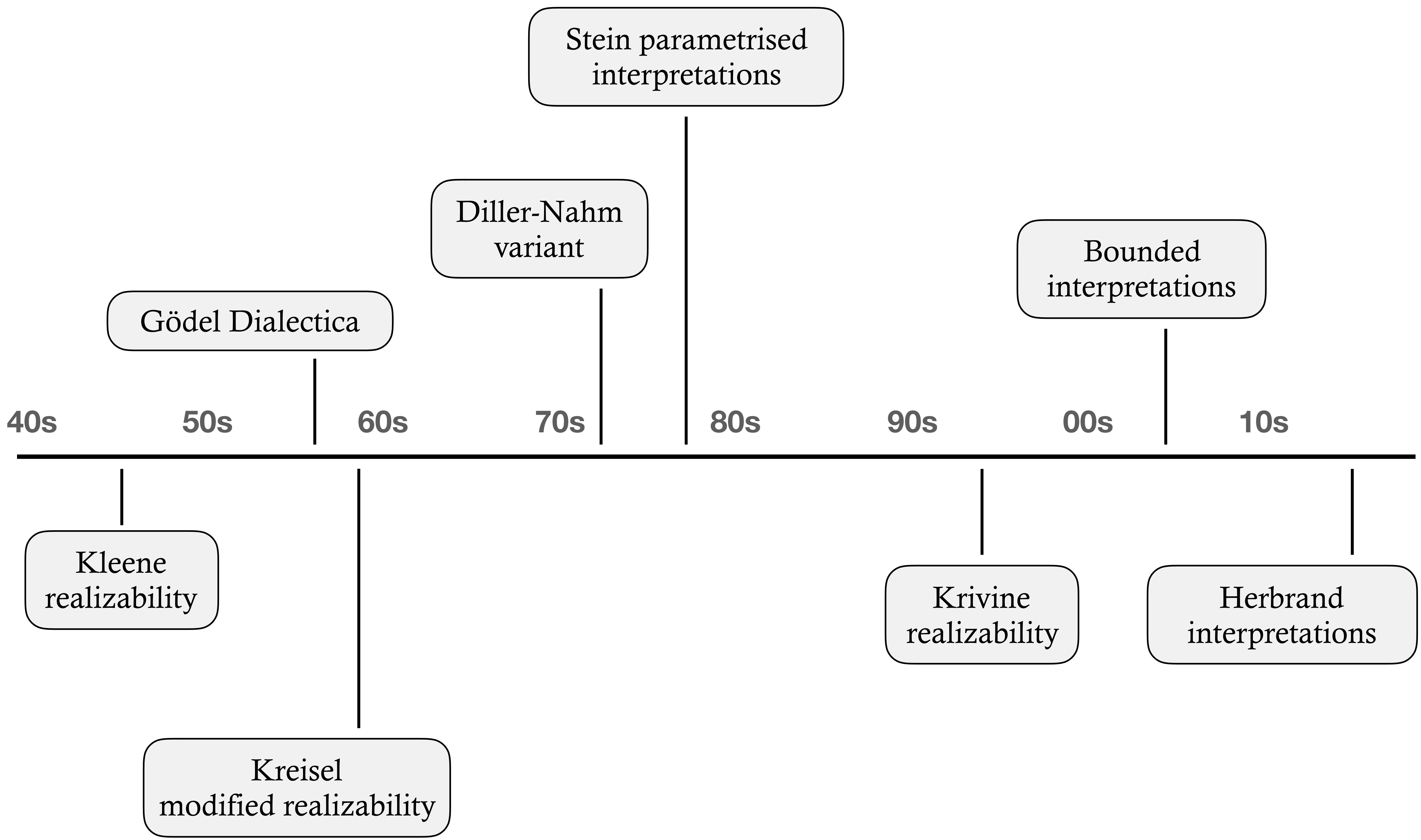
$$|A \vee B|^{x,v,b}_{y,w} \equiv (b{=}0 \wedge |A|^x_y) \vee (b \neq 0 \wedge |B|^v_w)$$

$$|A \to B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \to |B|^{f(x)}_w$$

$$|\forall z A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z A(z)|^{x,z}_y \equiv |A(z)|^x_y$$

Theorem (Gödel'58). If HA $\vdash A$ then there exists a term t of system T such that T $\vdash \forall y |A|^t_y$



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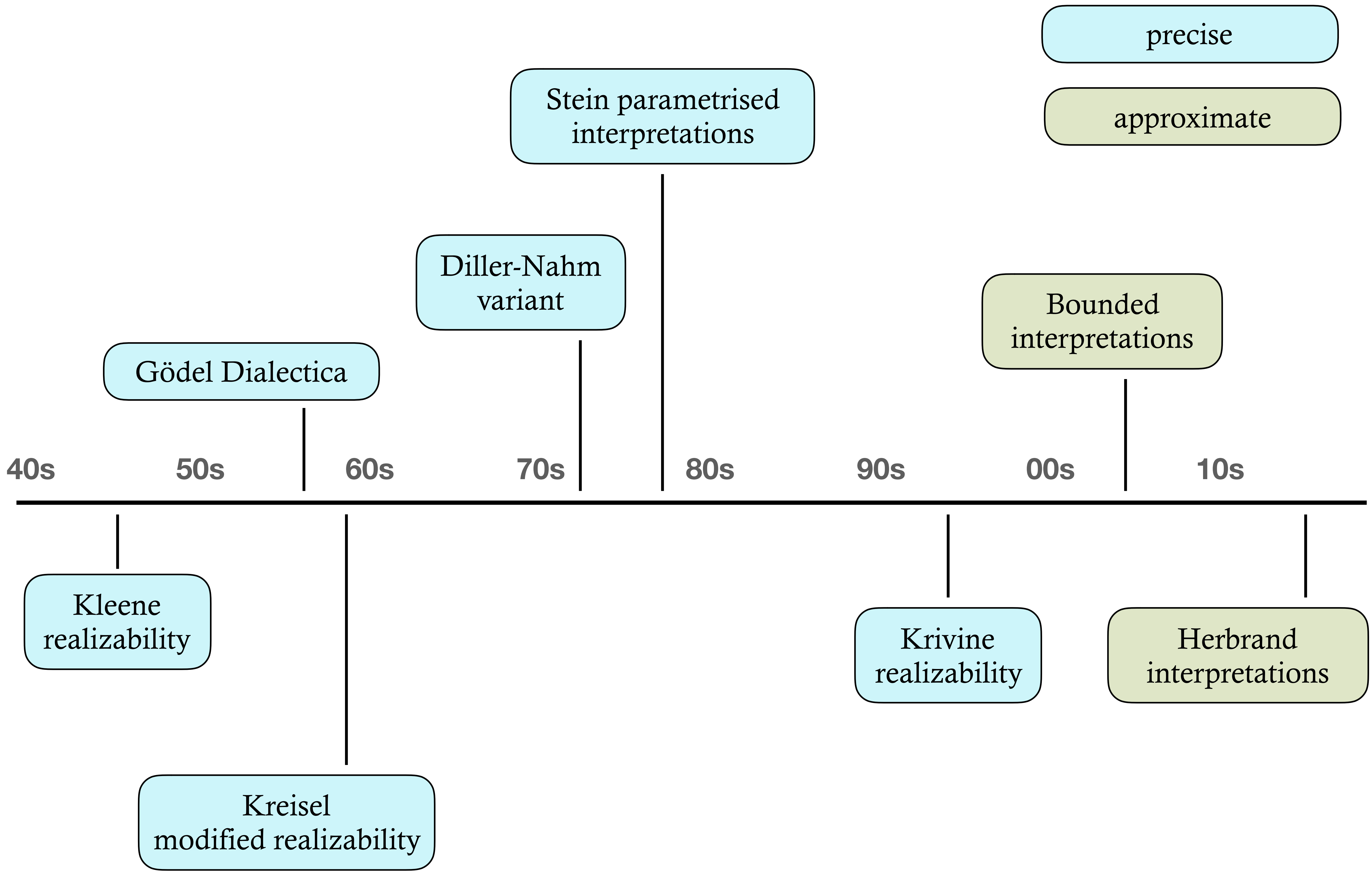
precise
approximate
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"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise

$$f(x) = \begin{cases} 0 & \text{if } \exists u. T(x,x,u) \\ 1 & \text{if } \forall u. \neg T(x,x,u) \end{cases}$$

precise

but

non-computable

Approximate

$$g(x) = \{0, 1\}$$

approximate

but

computable

precise inputs

precise outputs

non-computable inputs

non-computable outputs

approximate inputs

approximate outputs

computable inputs

computable outputs

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

“Herbrand” interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

What do they have in common?

In which way are they different?

Any others waiting to be discovered?

Unifying Functional Interpretations 1.0

Kleene realizability | Gödel Dialectica | Diller-Nahm variant | Bounded interpretations | "Herbrand" interpretations

Kreisel modified realizability | Krivine realizability | Stein parametrised interpretations

Result 1: Relational presentation of realizability

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

Result 2: Only differ in the treatment of contraction (!A)

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

Result 3: Multiple exponentials = combined interpretations

M.D. Hernest and P. Oliva, **Hybrid functional interpretations**, CiE, 2008

Linear Logic

A refinement of classical and intuitionistic logic

$$A \to B \quad \Rightarrow \quad !A \multimap B$$

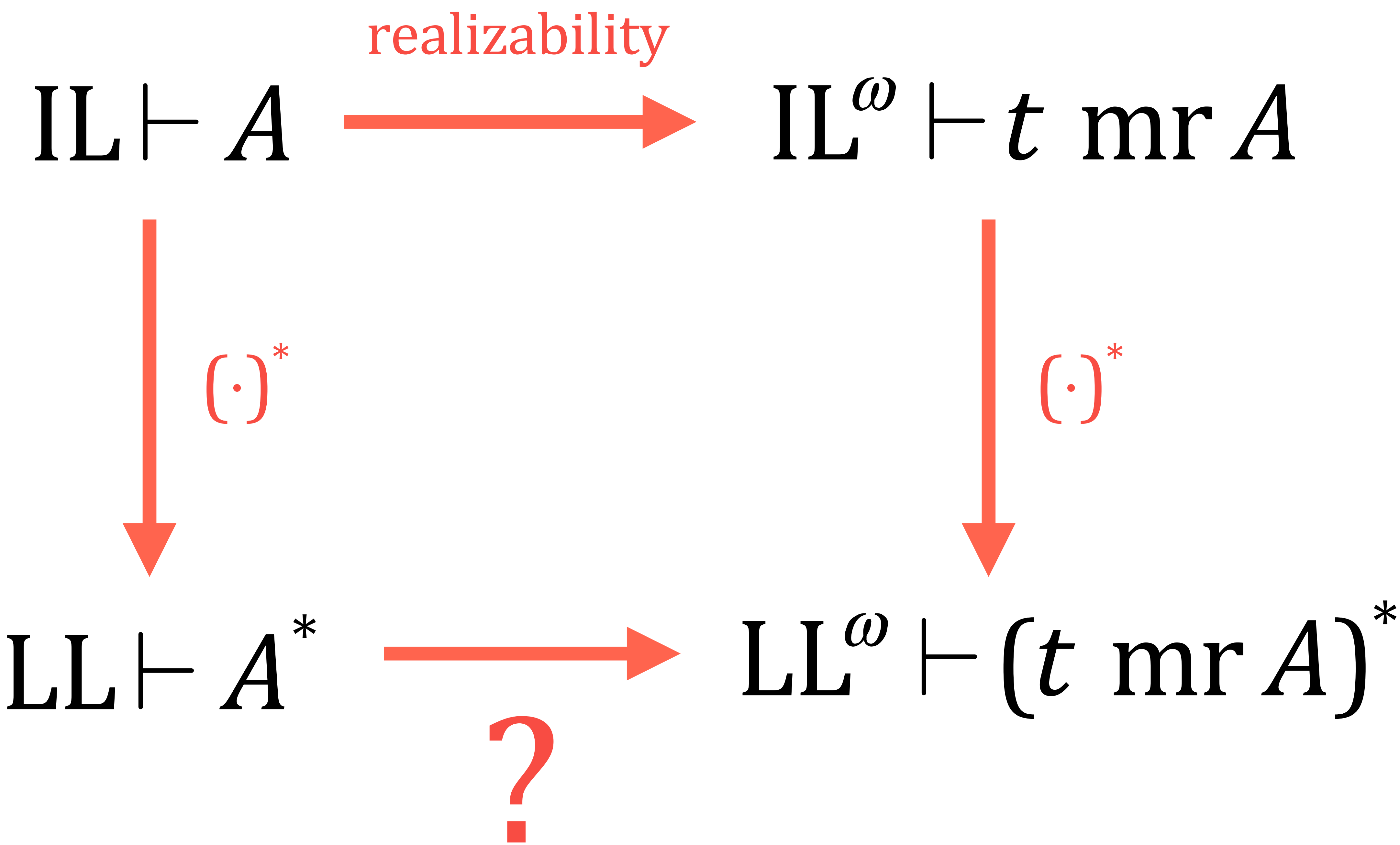
$$A \wedge B \quad \Rightarrow \quad A \,\&\, B$$

$$A \wedge B \quad \Rightarrow \quad A \otimes B$$

call-by-name translation		
$(A \wedge B)^*$	$\equiv$	$A^* \,\&\, B^*$
$(A \vee B)^*$	$\equiv$	$!A^* \oplus !B^*$
$(A \to B)^*$	$\equiv$	$!A^* \multimap B^*$
$(\forall z\, A)^*$	$\equiv$	$\forall z\, A^*$
$(\exists z\, A)^*$	$\equiv$	$\exists z !A^*$

call-by-value translation		
$(A \wedge B)^\circ$	$\equiv$	$A^\circ \otimes B^\circ$
$(A \vee B)^\circ$	$\equiv$	$A^\circ \oplus B^\circ$
$(A \to B)^\circ$	$\equiv$	$!(A^\circ \multimap B^\circ)$
$(\forall z\, A)^\circ$	$\equiv$	$!\forall z\, A^\circ$
$(\exists z\, A)^\circ$	$\equiv$	$\exists z\, A^\circ$

$$\mathrm{IL} \vdash A \quad \Rightarrow \quad \begin{array}{l} \mathrm{LL} \vdash A^\circ \\ \mathrm{LL} \vdash A^* \end{array}$$


realizability
$\mathrm{IL} \vdash A$
$\mathrm{IL}^\omega \vdash t \ \mathrm{mr} \ A$
$(\cdot)^*$
$(\cdot)^*$
$\mathrm{LL} \vdash A^*$
$\mathrm{LL}^\omega \vdash (t \ \mathrm{mr} \ A)^*$
?

Interpretation of Linear Logic

$$
\begin{aligned}
|A \otimes B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \otimes |B|^{v}_{w} \\
|A \oplus B|^{x,v,b}_{y,w} &\equiv (b{=}0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \,\&\, B|^{x,v}_{y,w,b} &\equiv (b{=}0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \multimap B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \multimap |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011 modified realizability

$A \longrightarrow x \text{ mr } A$

$(\cdot)^*$ $\downarrow$ $\qquad$ $\downarrow$ $(\cdot)^*$

$A^* \longrightarrow (x \text{ mr } A)^* \Leftrightarrow \forall y \, |A^*|^x_y$

$|!A|^x \equiv !\forall y \, |A|^x_y$

interpretations (only) differ in treatment of $!A$

Unifying Functional Interpretation 1.0

$$|A \otimes B|^{x,v}_{y,w} \equiv |A|^x_y \otimes |B|^v_w$$

$$|A \oplus B|^{x,v,b}_{y,w} \equiv (b=0 \,\&\, |A|^x_y) \oplus (b \neq 0 \,\&\, |B|^v_w)$$

$$|A \,\&\, B|^{x,v}_{y,w,b} \equiv (b=0 \,\&\, |A|^x_y) \oplus (b \neq 0 \,\&\, |B|^v_w)$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \multimap |B|^{f(x)}_w$$

$$|\forall z\, A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z\, A(z)|^{x,z}_y \equiv |A(z)|^x_y$$

$$|!A|^x_b \equiv \forall y \sqsubset b\, |A(z)|^x_y$$

parameter to deal with contraction

Parametrised interpretation of LL

$$
\begin{aligned}
|A \otimes B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \otimes |B|^{v}_{w} \\
|A \oplus B|^{x,v,b}_{y,w} &\equiv (b=0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \,\&\, B|^{x,v}_{y,w,b} &\equiv (b=0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \multimap B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \multimap |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y} \\
|!A|^{x}_{b} &\equiv \forall y \sqsubset b\, |A(z)|^{x}_{y}
\end{aligned}
$$

Parametrised interpretation of IL

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v,b}_{y,w} &\equiv (b=0 \rightarrow |A|^{x}_{y}) \wedge (b \neq 0 \rightarrow |B|^{v}_{w}) \\
|A \rightarrow B|^{f,g}_{x,w} &\equiv \forall y \sqsubset g(x,w)\, |A|^{x}_{y} \rightarrow |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

Translation of LL into IL

$$
\begin{aligned}
(A \wedge B)^{*} &\equiv A^{*} \,\&\, B^{*} \\
(A \vee B)^{*} &\equiv !A^{*} \oplus !B^{*} \\
(A \rightarrow B)^{*} &\equiv !A^{*} \multimap B^{*} \\
(\forall z\, A)^{*} &\equiv \forall z\, A^{*} \\
(\exists z\, A)^{*} &\equiv \exists z\, !A^{*}
\end{aligned}
$$

!A	Trans.	Interpretation
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Kreisel modified realizability
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$\lvert !A \rvert^x_a \equiv ! \lvert A \rvert^x_a$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\lvert !A \rvert^x \equiv !\forall y \lvert A \rvert^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$\lvert !A \rvert^x_a \equiv !\forall y \in a \lvert A \rvert^x_y \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

Unifying Functional Interpretations 2.0

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

“Herbrand” interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

How do these fit (if at all) into the “unification”?

Bounded functional interpretation

Fernando Ferreira[a,*], Paulo Oliva[b]

Annals of Pure and Applied Logic 135 (2005) 73–112

$$n \leq^* m \quad \equiv \quad n \leq m$$

$$f \leq^* g \quad \equiv \quad \forall a \forall x \leq^* a (fx \leq^* ga \wedge gx \leq^* ga)$$

$$|A \wedge B|^{x,v}_{y,w} \quad \equiv \quad |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v}_{y,w} \quad \equiv \quad (\forall y' \leq^* y |A|^x_{y'}) \vee (\forall w' \leq^* w |B|^v_{w'})$$

$$|A \to B|^{f,g}_{x,w} \quad \equiv \quad \forall y \leq^* g(x,w) |A|^x_y \to |B|^{f(x)}_w$$

$$|\forall z A(z)|^f_{y,c} \quad \equiv \quad \forall z \leq^* c |A(z)|^{f(c)}_y$$

$$|\exists z A(z)|^{x,c}_y \quad \equiv \quad \exists z \leq^* c \forall y' \leq y |A(z)|^x_{y'}$$

A functional interpretation for nonstandard arithmetic

Benno van den Berg[a,*,1], Eyvind Briseid[b,2], Pavol Safarik[c,3]

Annals of Pure and Applied Logic 163 (2012) 1962–1994

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^x_y \vee |B|^v_w$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g[x,w]\, |A|^x_y \to |B|^{f[x]}_w$$

$$|\forall z^{\mathrm{st}} A(z)|^f_{y,z} \equiv |A(z)|^{f[z]}_y$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_y \equiv \exists z' \in z \forall y' \in y\, |A(z')|^x_{y'}$$

$$f[x] := \bigcup_{f' \in f} f'(x)$$

Bounded functional interpretation

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv (\forall y' \leq^* y |A|^{x}_{y'}) \vee (\forall w' \leq^* w |B|^{v}_{w'})$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \leq^* g(x,w) |A|^{x}_{y} \to |B|^{f(x)}_{w}$$

$$|\forall z A(z)|^{f}_{y,c} \equiv \forall z \leq^* c |A(z)|^{f(c)}_{z}$$

$$|\exists z A(z)|^{x,c}_{y} \equiv \exists z \leq^* c \forall y' \leq y |A(z)|^{x}_{y'}$$

Herbrand functional interpretation

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g[x,w] |A|^{x}_{y} \to |B|^{f[x]}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \equiv |A(z)|^{f[z]}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \equiv \exists z' \in z \forall y' \in y |A(z')|^{x}_{y'}$$

Result: Herbrand interpretations have alternative presentations that use usual functional application

Herbrand functional interpretation (alternative presentation)

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g(x,w)\, |A|^{x}_{y} \to |B|^{f(x)}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \equiv |A(z)|^{f(z)}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \equiv \exists z' \in z\, \forall y' \in y\, |A(z')|^{x}_{y'}$$

Consequence 1: Arguments are not always sets

Consequence 2: Full monotonicity property no longer holds

Unifying Interpretation 2.0

$$|P(x)|^{a} \equiv x \prec_P a$$

$$|A \otimes B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \otimes |B|^{v}_{w}$$

$$|A \lozenge_z B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \lozenge_z |B|^{v}_{w}$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^{x}_{g(x,w)} \multimap |B|^{f(x)}_{w}$$

$$|\forall z A(z)|^{x}_{y} \equiv \forall z |A(z)|^{x}_{y}$$

$$|\exists z A(z)|^{x}_{y} \equiv \exists z |A(z)|^{x}_{y}$$

$$|!A|^{x}_{a} \equiv \forall y \sqsubset a |A|^{x}_{y}$$

one parameter for each atomic formula P

one parameter to deal with contraction

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\forall x^{\tau} A \equiv \forall x(\tau(x) \to A)$$
$$\exists x^{\tau} A \equiv \exists x(\tau(x) \wedge A)$$

Precise

$$|\tau(x)|^{a} \equiv x = a$$

Approximate

$$|\tau(x)|^{a} \equiv x \leq^{*} a$$

$$|\tau(x)|^{a} \equiv x \in a$$

...

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\forall x^{\tau,\mathrm{st}} A \quad \equiv \quad \forall x(\tau(x) \to \mathrm{st}(\mathrm{x}) \to A)$$
$$\exists x^{\tau,\mathrm{st}} A \quad \equiv \quad \exists x(\tau(x) \wedge \mathrm{st}(\mathrm{x}) \wedge A)$$

Type predicate / Standardness predicate

$$|\tau(x)| \quad \equiv \quad \tau(x)$$
$$|\mathrm{st}(x)|^a \quad \equiv \quad x \in a$$

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Definition 3.1 (**IL**-interpretations). *Given an* **AL**-*interpretation* $A \mapsto |A|^x_y$ *based on the translated parameters we can derive two* **IL**-*interpretations, namely*

$$A \quad \mapsto \quad (|A^*|^x_y)^{\mathcal{F}} \qquad \textit{and} \qquad A \quad \mapsto \quad (|A^\circ|^x_y)^{\mathcal{F}}$$

We will abbreviate these compound interpretations as

$$\{\!\{A\}\!\}^x_y \;\equiv\; (|A^*|^x_y)^{\mathcal{F}} \qquad \textit{and} \qquad (\!(A)\!)^x_y \;\equiv\; (|A^\circ|^x_y)^{\mathcal{F}}$$

Proposition 3.3. *Given an* **AL** *interpretation* $A \mapsto |A|^x_y$ *of* $\mathcal{A}_{\mathbf{s}}$ *into* $\mathcal{A}_{\mathbf{t}}$, *we have that the* **IL** *interpretation* $A \mapsto \{\!\{A\}\!\}^x_y$ *of* $\mathcal{A}^{\mathcal{F}}_{\mathbf{s}}$ *into* $\mathcal{A}^{\mathcal{F}}_{\mathbf{t}}$ *satisfies (provably in* $\mathcal{A}^{\mathcal{F}}_{\mathbf{t}}$ *)*

$$
\begin{array}{lcl}
\{\!\{P(\boldsymbol{x})\}\!\}^a & \equiv & \boldsymbol{x} \prec^P a \quad \textit{if } P \in \mathbf{Pred}^c_{\mathcal{A}_{\mathbf{s}}} \\
\{\!\{P(\boldsymbol{x})\}\!\} & \equiv & P(\boldsymbol{x}) \quad \textit{if } P \in \mathbf{Pred}^{nc}_{\mathcal{A}_{\mathbf{s}}} \\
\{\!\{A \to B\}\!\}^{f,g}_{x,w} & \equiv & \forall \boldsymbol{y} \sqsubset \boldsymbol{f x w}\, \{\!\{A\}\!\}^x_y \to \{\!\{B\}\!\}^{gx}_w \\
\{\!\{A \wedge B\}\!\}^{x,v}_{y,w} & \equiv & \{\!\{A\}\!\}^x_y \wedge \{\!\{B\}\!\}^v_w \\
\{\!\{A \vee B\}\!\}^{b,x,v}_{y,w} & \equiv & \exists z \prec^{\mathbb{B}} b((z = \mathrm{T} \to \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, \{\!\{A\}\!\}^x_{y'}) \wedge (z = \mathrm{F} \to \forall \boldsymbol{w}' \sqsubset \boldsymbol{w}\, \{\!\{B\}\!\}^v_{w'})) \\
\{\!\{\exists z A\}\!\}^x_y & \equiv & \exists z \forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, \{\!\{A\}\!\}^x_{y'} \\
\{\!\{\forall z A\}\!\}^x_y & \equiv & \forall z \{\!\{A\}\!\}^x_y
\end{array}
$$

In particular, we have that for computational predicate symbols P:

$$\{\!\{\exists z^P A\}\!\}^{c,\boldsymbol{x}}_{\boldsymbol{y}} \equiv \exists z <^P c \forall \boldsymbol{y}' \sqsubset \boldsymbol{y} \{\!\{A\}\!\}^{\boldsymbol{x}}_{\boldsymbol{y}'}$$

$$\{\!\{\forall z^P A\}\!\}^{f}_{b,\boldsymbol{y}} \equiv \forall z <^P b \{\!\{A\}\!\}^{fb}_{\boldsymbol{y}}$$

Bounded functional interpretation

$$|\exists z^\tau A(z)|^{c,x}_{y} \equiv \exists z \leq^* c \forall y' \leq^* y \, |A(z)|^{x}_{y'}$$

$$|\forall z^\tau A(z)|^{f}_{c,y} \equiv \forall z \leq^* c \, |A(z)|^{f(c)}_{z}$$

Herbrand functional interpretation

$$|\exists z^{\mathrm{st}_\tau} A(z)|^{c,x}_{y} \equiv \exists z \in c \forall y' \in y \, |A(z)|^{x}_{y'}$$

$$|\forall z^{\mathrm{st}_\tau} A(z)|^{f}_{c,y} \equiv \forall z \in c \, |A(z)|^{f(c)}_{y}$$

$\forall y \sqsubset a \|A\|^x_y$	$x \prec_P a$	**Interpretation**
$\forall y \|A\|^x_y$	$\tau(x) \wedge (x = a)$	Kreisel modified realizability
$\forall y \in a \|A\|^x_y$	$\tau(x) \wedge (x = a)$	Diller-Nahm interpretation
$\|A\|^x_a$	$\tau(x) \wedge (x = a)$	Gödel's Dialectica interpretation
$\forall y \|A\|^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded modified realizability
$\forall y \leq^* a \|A\|^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded functional interpretation
$\forall y \|A\|^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand modified realizability
$\forall y \in a \|A\|^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand functional interpretation

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

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J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

P. Oliva, **Kreisel's modified realizability and recent variants**, to appear

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation