Tutorial on Proof Theory

(with emphasis on proof mining)

Lecture 1: Formal Proofs

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Proof of A $\longrightarrow$ Truth of A

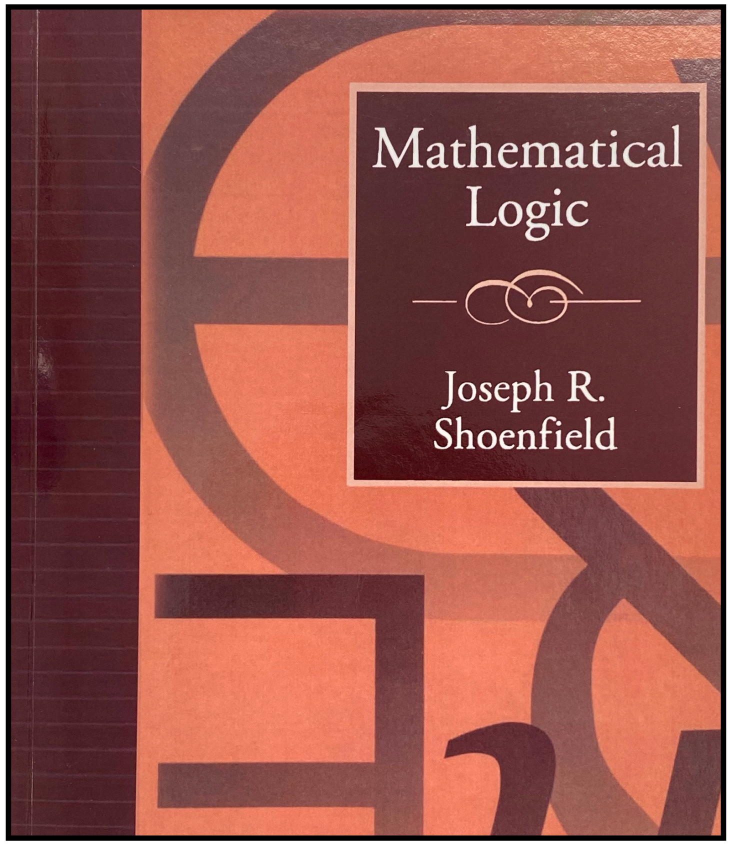
Construction for A $\longrightarrow$ Truth of A

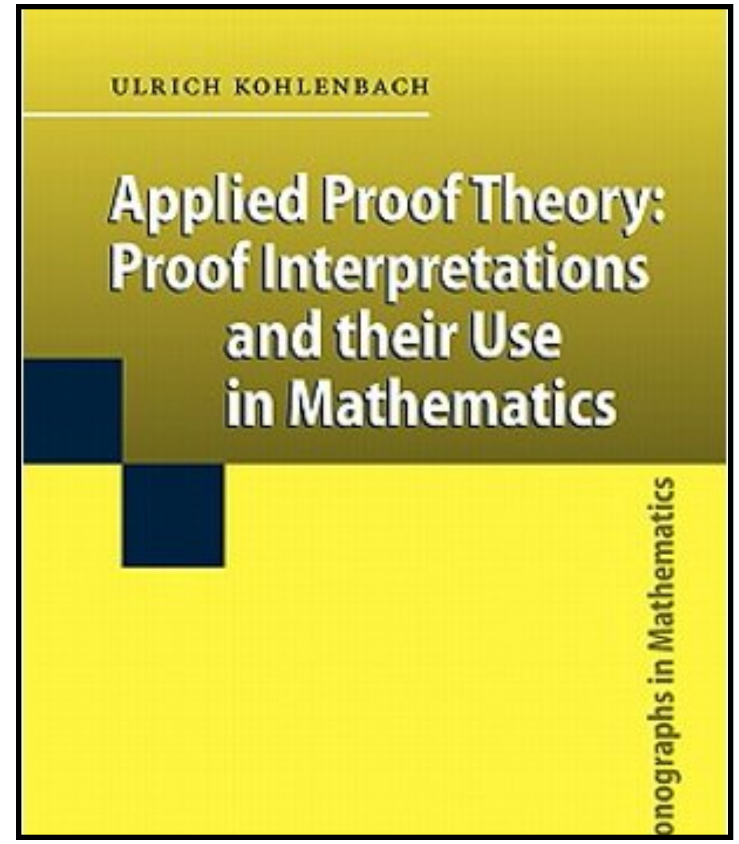
No proof of 0=1 $\longrightarrow$ Consistency

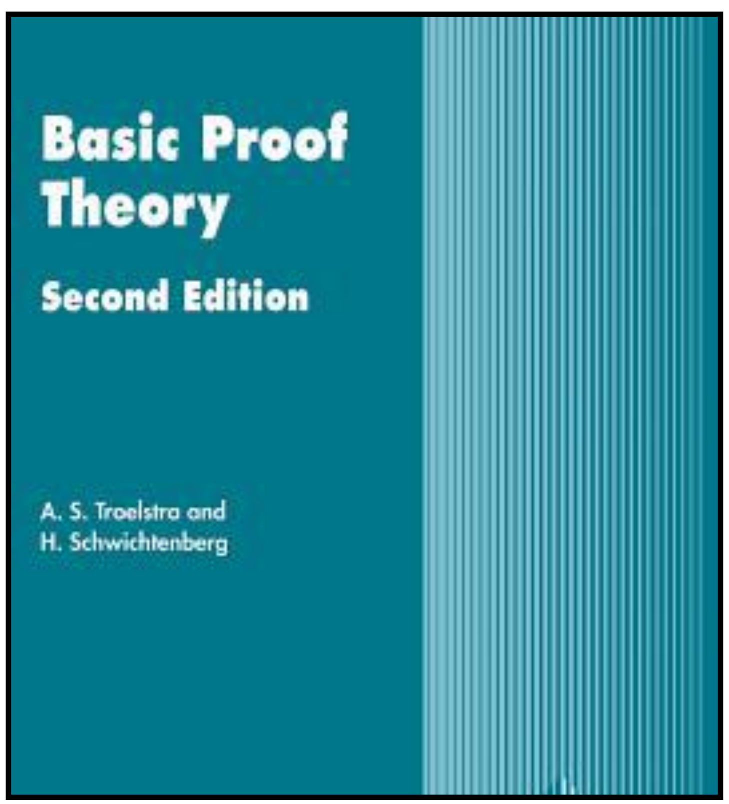
No short proof of A_n $\longrightarrow$ Separation of complexity classes

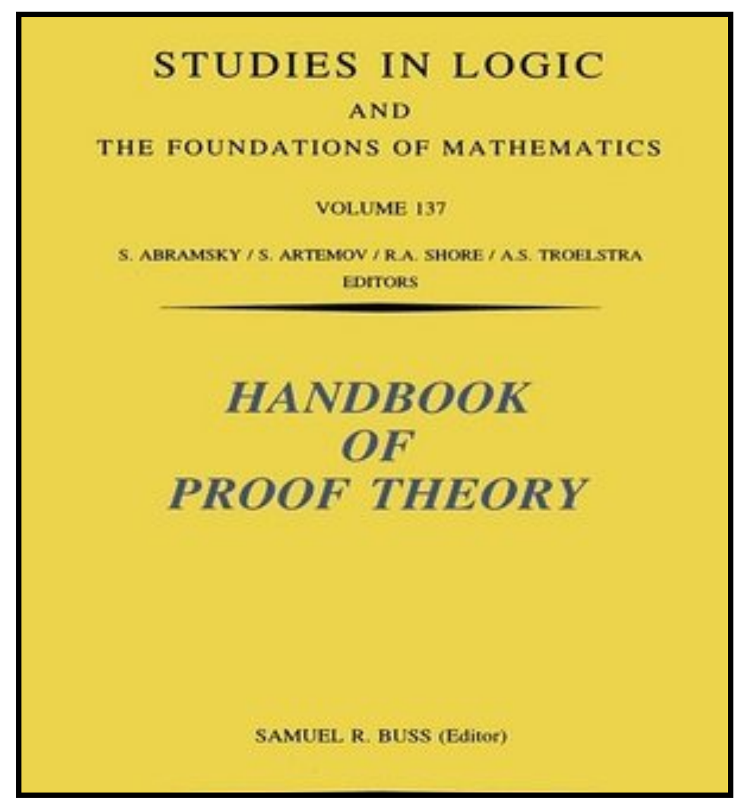
Proof of A $\longrightarrow$ Program of type A

Proof of A $\longrightarrow$ More than truth of A


Mathematical Logic
Joseph R. Shoenfield


ULRICH KOHLENBACH
Applied Proof Theory: Proof Interpretations and their Use in Mathematics
onographs in Mathematics


Basic Proof Theory
Second Edition
A. S. Troelstra and
H. Schwichtenberg


STUDIES IN LOGIC
AND
THE FOUNDATIONS OF MATHEMATICS
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S. ABRAMSKY / S. ARTEMOV / R.A. SHORE / A.S. TROELSTRA
EDITORS
HANDBOOK OF PROOF THEORY
SAMUEL R. BUSS (Editor)

Plan

Lecture 1: **Formal Proofs**

Lecture 2: Proof Translations

Lecture 3: Proof Interpretations

Today

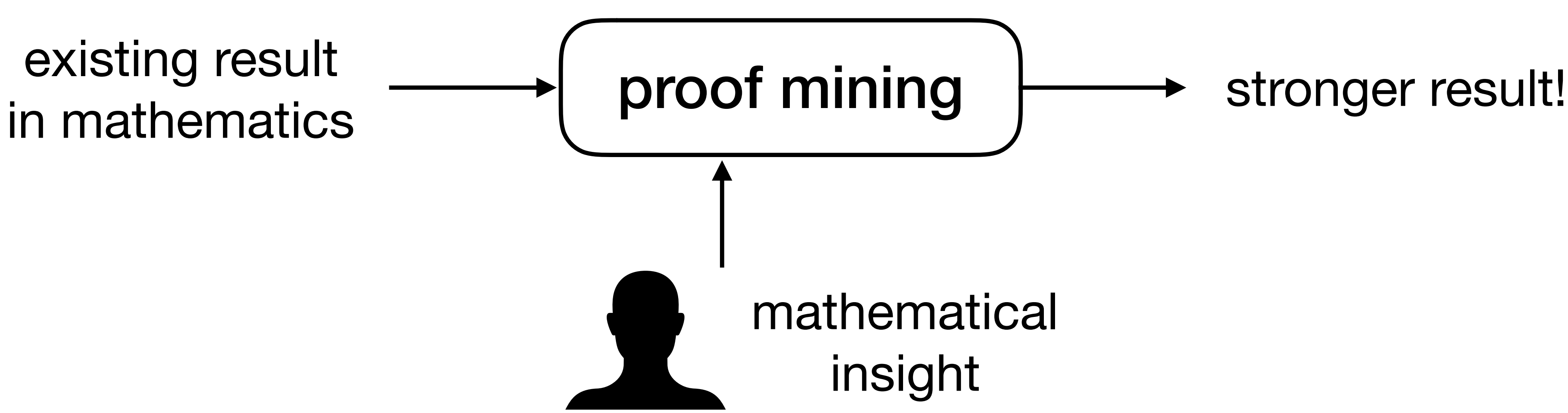
- Applied proof theory: Proof mining
- Formal language
- Formal proofs
- Some examples!

Proof Mining

Proof Mining

research program to obtain extra information from (nonconstructive) mathematical proofs

- Originated with Kreisel's applications of his "no-counterexample interpretation"
- Resurgence in 1990s with Kohlenbach's application of his "monotone functional interpretation"



Proof Mining

some success stories...

- [1993] Chebyshev approximation of functions by polynomials
- [2001] L_1 approximation of functions by polynomials
- [2001-3] Krasnoselski fixed point theorem (nonexpansive maps on normed and hyperbolic spaces)
- [2009] Mean ergodic theorem for Banach spaces
- [2011] Browder/Wittmann fixed point theorems (nonexpansive maps on Hilbert spaces)
- [2012-6] Generalisations for CAT(0) and CAT(k) spaces
- [2019] Generalisations for smooth/convex Banach spaces

$$\sqrt{2} \notin \mathbb{Q}$$

Theorem. $\sqrt{2} \notin \mathbb{Q}$

What more can we say about this theorem?

Proof.

Assume we have $p,\, q \in \mathbb{N}$ such that $\dfrac{p}{q} = \sqrt{2}$

W.l.g., we can assume that $p,\, q \in \mathbb{N}$ are relatively prime

Then $\dfrac{p^2}{q^2} = 2$, and hence $p^2 = 2q^2$, so p must be even

Let $p = 2a$. Then $4a^2 = 2q^2$, and hence $2a^2 = q^2$, so q must be even

This contradicts the assumption that p,q are relatively prime. $\square$

What extra information does this proof carry?

Theorem A. $\sqrt{2} \notin \mathbb{Q}$

Theorem B. For all $p, q \in \mathbb{N}$ with $q > 0$, if $p / q = \sqrt{2}$ then p, q are even

Theorem C. For all $p, q \in \mathbb{N}$ with $q > 0$, if either p or q is not even then $p / q \neq \sqrt{2}$

Theorem D. For all $p, q \in \mathbb{N}$ with $q > 0$, if either p or q is not even then $|p / q - \sqrt{2}| > \delta$, for some $\delta > 0$

Theorem D. For all $p,q \in \mathbb{N}$ with $q > 0$, if either p or q is not even then $|p/q - \sqrt{2}| > \delta$, for some $\delta > 0$

Theorem E. For all $p,q > 0$ with p or q not even, we have

$$|\frac{p}{q} - \sqrt{2}| > \frac{1}{pq + 2q^2}$$

Lemma. If $x, y > 0$ and $|x^2 - y^2| \geq \delta$ then $|x - y| \geq \delta / (x + y)$

Proof. Follows from $(x^2 - y^2) = (x + y)(x - y)$. $\square$

Lemma. If $x, y > 0$ and $|x^2 - y^2| \geq \delta$ then $|x - y| \geq \delta / (x + y)$

Theorem E. For all $p,q > 0$ with p or q not even, we have

$$|\frac{p}{q} - \sqrt{2}| > \frac{1}{pq + 2q^2}$$

Proof. Fix $p,\ q > 0$ and assume they are not both even.

It follows that $p^2 \neq 2q^2$ and $|p^2 - 2q^2| \geq 1$

Hence $|p^2 / q^2 - 2| \geq 1 / q^2$, and by the lemma above

$$|\frac{p}{q} - \sqrt{2}| \geq \frac{1}{q(p + q\sqrt{2})} > \frac{1}{pq + 2q^2} \quad \square$$

Theorem. $\sqrt{2} \notin \mathbb{Q}$

?

1. What strengthening of the theorem is possible?

2. How to obtain strengthening from the proof?

3. What extra lemmas are needed?

Theorem. For all $p, q > 0$ with p or q not even, we have

$$\left|\frac{p}{q} - \sqrt{2}\right| > \frac{1}{pq + 2q^2}$$

Formal Language

Atomic formulas

$\perp$ (contradiction)

$n \in \mathbb{N}, x \in \mathbb{R}, \ldots$

$n =_{\mathbb{N}} m, n \leq_{\mathbb{N}} m, \ldots$

Connectives

$A \wedge B$ (A and B)

$A \vee B$ (A or B)

$A \to B$ (A implies B)

Quantifiers

$\forall x\, A$ (A holds for all x)

$\exists x\, A$ (A holds for some x)

Abbreviations

$\neg A \equiv A \to \perp$

$\forall n^{\mathbb{N}} A(n) \equiv \forall n (n \in \mathbb{N} \to A(n))$

$\exists x^{\mathbb{R}} A(n) \equiv \exists x (x \in \mathbb{R} \wedge A(n))$

$x \in \mathbb{Q}^{+} \equiv x \in \mathbb{Q} \wedge (x > 0)$

Examples

The function $f : \mathbb{R} \to \mathbb{R}$ is continuous

$$\forall x^{\mathbb{R}}, \varepsilon^{\mathbb{Q}^+} \exists \delta^{\mathbb{Q}^+} \forall y^{\mathbb{R}} (|x - y| <_{\mathbb{R}} \delta \to |fx - fy| <_{\mathbb{R}} \varepsilon)$$

The function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous

$$\forall \varepsilon^{\mathbb{Q}^+} \exists \delta^{\mathbb{Q}^+} \forall x^{\mathbb{R}}, y^{\mathbb{R}} (|x - y| <_{\mathbb{R}} \delta \to |fx - fy| <_{\mathbb{R}} \varepsilon)$$

The sequence $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}$ converges to $x \in \mathbb{R}$

$$\forall \varepsilon^{\mathbb{Q}^+} \exists n^{\mathbb{N}} \forall m \geq n \, (|a_m - x| <_{\mathbb{R}} \varepsilon)$$

First-order Logic

(natural deduction system)

Introduction Rules

$$\frac{\dfrac{\vdots}{A} \qquad \dfrac{\vdots}{B}}{A \wedge B}\ \wedge\mathrm{I}$$

$$\frac{\dfrac{\vdots}{A}}{A \vee B}\ \vee\mathrm{I} \qquad \frac{\dfrac{\vdots}{B}}{A \vee B}\ \vee\mathrm{I}$$

$$\frac{\dfrac{\dfrac{[A]_\alpha}{\vdots}}{B}}{A \to B}\ \to\mathrm{I},\alpha$$

Elimination Rules

$$\frac{\dfrac{\vdots}{A \wedge B}}{A}\ \wedge\mathrm{E} \qquad \frac{\dfrac{\vdots}{A \wedge B}}{B}\ \wedge\mathrm{E}$$

$$\frac{\dfrac{\vdots}{A \vee B} \qquad \dfrac{\dfrac{[A]_\alpha}{\vdots}}{C} \qquad \dfrac{\dfrac{[B]_\beta}{\vdots}}{C}}{C}\ \vee\mathrm{E},\alpha,\beta$$

$$\frac{\dfrac{\vdots}{A} \qquad \dfrac{\vdots}{A \to B}}{B}\ \to\mathrm{E}$$

Introduction Rules

$$\dfrac{\dfrac{\dfrac{\Gamma}{\vdots}}{A(x)}}{\forall x\, A(x)}\quad x \notin \mathrm{FV}(\Gamma)$$

$$\dfrac{\dfrac{\vdots}{A(t)}}{\exists x\, A(x)}$$

Elimination Rules

$$\dfrac{\dfrac{\vdots}{\forall x\, A(x)}}{A(t)}$$

$$\dfrac{\dfrac{\vdots}{\exists x\, A(x)} \qquad \dfrac{\dfrac{[A(x)]}{\vdots}}{C}}{C}\quad x \notin \mathrm{FV}(C)$$

The just system described is called Minimal Logic **ML**

Ex falso quodlibet

$$\dfrac{\begin{matrix}\vdots \\ \hline \bot\end{matrix}}{A}\ \mathrm{EFQ}$$

ML + **EFQ** is called intuitionistic logic **IL**

Proof by contradiction

$$\dfrac{\begin{matrix}\dfrac{[\neg A]_\alpha}{\vdots} \\ \hline \bot\end{matrix}}{A}\ \mathrm{PBC},\ \alpha$$

ML + **PBC** is called classical logic **CL**

Tree Style

Sequent Style

$$\dfrac{\dfrac{\vdots}{A} \qquad \dfrac{\vdots}{B}}{A \wedge B}\,\wedge\mathrm{I} \quad \longrightarrow \quad \dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge\mathrm{I}$$

$$\dfrac{\dfrac{\vdots}{A \vee B} \qquad \dfrac{\dfrac{[A]_\alpha}{\vdots}}{C} \qquad \dfrac{\dfrac{[B]_\beta}{\vdots}}{C}}{C}\,\vee\mathrm{E}, \alpha, \beta \quad \longrightarrow \quad \dfrac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\,\vee\mathrm{E}$$

$$\dfrac{\dfrac{\dfrac{[A]_\alpha}{\vdots}}{B}}{A \to B}\,\to\mathrm{I}, \alpha \quad \longrightarrow \quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\to\mathrm{I}$$

Aside: Linear Logic

$$\dfrac{\dfrac{\overline{[A]_\alpha}}{\vdots}}{\dfrac{\overline{\quad B \quad}}{A \to B}} \to \mathrm{I}, \alpha$$

In classical/ intuitionistic/minimal logic one doesn't "count" the number of times A appears as an assumption

First-order logic

assumes contraction

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\mathrm{I}$$

same assumption used multiple times

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \rightarrow\mathrm{E}$$

Linear Logic

multiplicative conjunction

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes\mathrm{I}$$

additive conjunction

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \&\mathrm{I}$$

linear implication

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A \multimap B}{\Gamma, \Delta \vdash B} \multimap\mathrm{E}$$

Formal Proofs

$$\boxed{\neg\neg A,\ \neg\neg B\ \vdash\ \neg\neg(A\wedge B)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha \quad [B]_\beta}{A \wedge B} \quad [\neg(A\wedge B)]_\gamma}{\bot}}{\neg A}\,\alpha \qquad \neg\neg A}{\bot}}{\neg B}\,\beta \qquad \neg\neg B}{\bot}}{\neg\neg(A\wedge B)}\,\gamma$$

$$\boxed{\vdash A \vee \neg A}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\bot}}{\neg A}\,\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\dfrac{\bot}{A \vee \neg A}\ \text{PBC}, \gamma}$$

Theorem. $\sqrt{2} \notin \mathbb{Q}$

Proof.

Assume we have $p,\ q \in \mathbb{N}$ such that $\dfrac{p}{q} = \sqrt{2}$

W.l.g., we can assume that $p,\ q \in \mathbb{N}$ are relatively prime

Then $\dfrac{p^2}{q^2} = 2$, and hence $p^2 = 2q^2$, so p must be even

Let $p = 2a$. Then $4a^2 = 2q^2$, and hence $2a^2 = q^2$, so q must be even

This contradicts the assumption that p,q are relatively prime. $\square$

assumption used twice (contraction)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p/q=\sqrt{2}]_\alpha}{p=q\sqrt{2}}}{p^2=2q^2}}{\exists a(p=2a)} \qquad [p=2a] \qquad \dfrac{\dfrac{[p/q=\sqrt{2}]_\alpha}{p=q\sqrt{2}}}{p^2=2q^2}}{\exists a(p=2a)\wedge\exists b(q=2b)}\boxed{\exists\mathrm{E},\beta} \qquad [\forall a(p\neq 2a)\vee\forall b(q\neq 2b)]_\gamma}{\perp}}{p/q\neq\sqrt{2}}\boxed{\to\mathrm{I},\alpha}}{\forall a(p\neq 2a)\vee\forall b(q\neq 2b)\to p/q\neq\sqrt{2}}\boxed{\to\mathrm{I},\gamma}}{\forall p,q(\forall a(p\neq 2a)\vee\forall b(q\neq 2b)\to(p/q\neq\sqrt{2}))}\boxed{\forall\mathrm{I}}
$$

Tomorrow…

- Complete vs incomplete statements
- Intuitionistic vs classical proofs
- Double negation translations