Modified Realizability Interpretation of Classical Linear Logic

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Abstract

This paper presents a modified realizability interpretation of classical linear logic. The interpretation is based on work of de Paiva (1989), Blass (1995), and Shirahata (2006) on categorical models of classical linear logic using Gödel's Dialectica interpretation. Whereas the Dialectica categories provide models of linear logic, our interpretation is presented as an endo-interpretation of proofs, which does not leave the realm of classical linear logic. The advantage is that we obtain stronger versions of the disjunction and existence properties, and new conservation results for certain choice principles. Of particular interest is the simple branching quantifier used in order to obtain a completeness result for the modified realizability interpretation.

1 Introduction

This paper presents a modified realizability interpretation of classical linear logic. A completeness result is obtained for the interpretation with the help of a simple form of branching quantifier. Proof-theoretic applications such as closure and conservation properties are also discussed.

The realizability interpretation presented here is based on work of de Paiva [16, 17] and Shirahata [19] on the Dialectica interpretation [11] of linear logic. In fact, the treatment of all the connectives of linear logic follows [17] verbatim. The difference comes solely in the interpretation of the exponentials. The shift to the modified realizability comes from the author's work on the unifying framework for functionals interpretations [15] and Blass' comments at the end of [4] suggesting a simplification on de Paiva's work. Whereas de Paiva's work on the Dialectica interpretation aimed at the construction of categorical models of linear logic, our interpretation is presented as an endo-interpretation of proofs, which does not leave the realm of classical linear logic. The advantage is that we obtain stronger versions of the disjunction and existence properties for extensions of classical linear logic, and new conservation results for certain choice principles.

Intuitively, functional interpretations such as the Dialectica interpretation [11, 17] and Kreisel's modified realizability [14] associate formulas with one-move games between two players ($\exists$loise and $\forall$belard) and proofs with winning strategies for $\exists$loise. The interpretation of each of the logical connectives, quantifiers and exponentials corresponds to constructions that build new games out of given games. In the case of classical linear logic, the interpretation is totally symmetric with respect to linear negation, so that the game corresponding to $A^{\perp}$ is the game A with the roles of the two players swapped. The modified realizability presented here interprets the exponentials as games where only one player needs to make a move. For instance, in the game $?A$, only $\forall$belard makes a move, and $\exists$loise will win in case she has a winning move for the game A with the given $\forall$belard's move. A symmetric situation occurs in the case of the game $!A$, only that $\forall$belard now has the advantage. The idea is that the exponentials $?$ and $!$ serve as trump cards for $\exists$loise and $\forall$belard, respectively.

Researchers familiar with linear logic will no doubt be also familiar with its constructive aspect. For those, the possibility of a realizability interpretation of classical linear logic may come as no surprise. For proof theorists less familiar with linear logic, however, it may seem hard to believe that modified realizability can be directly applied to a classical system, given that modified realizability interpretations of classical systems are normally only possible via an initial embedding of the classical system into an intuitionistic variant. Moreover, that often needs to be followed by Friedman's A-translation in order to eliminate double negations in front of Σ^0_1-formulas, if one is interested in the provably total functions of the system. As we will see, none of this is necessary in the case of classical linear logic.

The paper is organised as follows. The modified realiz-

$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes)$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap)$	$\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(\text{cut})$	$A_{\mathsf{at}} \vdash A_{\mathsf{at}} \quad (\text{id})$
$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash\, !A}\,(!)$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash\, ?A}\,(?)$	$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,(\text{con})$	$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,(\text{wkn})$
$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall z^\rho A}\,(\forall)$	$\dfrac{\Gamma \vdash A[t^\rho/z]}{\Gamma \vdash \exists z^\rho A}\,(\exists)$	$\dfrac{\Gamma, A \vdash B}{\Gamma, B^\perp \vdash A^\perp}\,(\perp)$	$\dfrac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\,(\text{per})$

Table 1. Classical linear logic LL^ω (additives treated in Section 5)

ability interpretation of linear logic is presented in Section 2. Soundness of the interpretation is proved in Section 3. Completeness of the interpretation is presented in Section 4. A simple form of branching quantifier is used for the proof of completeness. Due to the special way the additives are treated, their interpretation is discussed in Section 5. In Section 6, we show how the realizability interpretation of classical linear logic relates to Kreisel's [14] modified realizability interpretation of intuitionistic logic, via Girard's embedding of IL into LL. Other variants of the modified realizability interpretation are discussed in Section 7, including Gödel's Dialectica interpretation of linear logic [16, 19]. In the final section, we discuss the semantical aspect of the interpretation and related work.

For an introduction to modified realizability, see chapter III of [20] or the book chapter [21]. For an introduction to linear logic, see Girard's original papers [9, 10].

1.1 Classical Linear Logic LL^ω

We work with an extension of classical linear logic to the language of all finite types. The set of *finite types* $\mathcal{T}$ is inductively defined as follows:

- $o \in \mathcal{T}$;
- if $\rho, \sigma \in \mathcal{T}$ then $\rho \to \sigma \in \mathcal{T}$.

For simplicity, we deal with only one basic finite type o.

Remark 1.1 (Finite types extension) *The extension of the language of linear logic to all finite types is not necessary for the system under interpretation, but is required in the "verifying system", i.e. we could have presented an interpretation of* LL *(pure classical linear logic) into* LL^ω*, but we chose to start already with the system* LL^ω *so that the interpretation is an endo-interpretation.*

We assume that the terms of LL^ω contain all typed λ-terms, i.e. variables x^ρ for each finite type ρ; λ-abstractions $(\lambda x^\rho . t^\sigma)^{\rho\to\sigma}$; and term applications $(t^{\rho\to\sigma} s^\rho)^\sigma$. Note that we work with the standard typed λ-calculus, and not with a linear variant thereof. The atomic formulas of LL^ω are $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^\perp, B_{\mathsf{at}}^\perp, \ldots$. For simplicity, the standard propositional constants $0, 1, \perp, \top$ of linear logic have been omitted, since the realizability interpretation of atomic formulas is trivial (see Definition 2.1).

The *linear negation* $A^\perp$ of an arbitrary formula A is an abbreviation as follows:

$$
\begin{array}{lcllcl}
(A_{\mathsf{at}})^\perp & \equiv & A_{\mathsf{at}}^\perp & (\exists z A)^\perp & \equiv & \forall z A^\perp \\
(A_{\mathsf{at}}^\perp)^\perp & \equiv & A_{\mathsf{at}} & (\forall z A)^\perp & \equiv & \exists z A^\perp \\
(A \multimap B)^\perp & \equiv & A \otimes B^\perp & (?A)^\perp & \equiv & !(A^\perp) \\
(A \otimes B)^\perp & \equiv & A \multimap B^\perp & (!A)^\perp & \equiv & ?(A^\perp).
\end{array}
$$

So, $(A^\perp)^\perp$ is syntactically equal to A. We will often write $\vdash A \leftrightarrow B$ as a shorthand for the fact that both $A \multimap B$ and $B \multimap A$ are provable.

The rules for classical linear logic are shown in Table 1, with the usual side condition in the rule $(\forall)$ that the variable z must not appear free in Γ. We postpone the treatment of the additives to Section 5, since our treatment of these is rather unconventional.

Our formulation of the rules of classical linear logic differs slightly from Girard's original formulation in [10]. First, we use the linear implication $A \multimap B$ rather than the multiplicative disjunction $A \parr B$, which can be defined as $A \parr B \equiv A^\perp \multimap B$. Moreover, we use two-sided sequents, with only one formula on the right side, simply to mark the principal formula of the sequent. If the reader prefers, our sequent $A_0, \ldots, A_n \vdash B$ can be read as $\vdash A_0^\perp, \ldots, A_n^\perp, B$.

Notation 1.2 *We use bold face variables $\boldsymbol{f}, \boldsymbol{g}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \ldots$ for tuples of variables, and bold face terms $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{\gamma}, \boldsymbol{\delta}, \ldots$ for tuples of terms. Given sequence of terms $\boldsymbol{a}$ and $\boldsymbol{b}$, by $\boldsymbol{a}(\boldsymbol{b})$, we mean the sequence of terms $a_0(\boldsymbol{b}), \ldots, a_n(\boldsymbol{b})$. Similarly for $\boldsymbol{a}[\boldsymbol{b}/\boldsymbol{x}]$.*

2 Realizability Interpretation of LL^ω

In this section we present the realizability interpretation of classical linear logic LL^ω. The interpretation is similar to de Paiva's Dialectica interpretation [16, 17] save the treatment of the exponentials. We discuss this in more details in Section 7.

To each formula A of linear logic we associate a formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, with two fresh sequences of free-variables $\boldsymbol{x}, \boldsymbol{y}$. The tuple of variables $\boldsymbol{x}$ in the superscript are called the *witnessing variables*, while the subscript variables $\boldsymbol{y}$ are called the *challenge variables*[1]. Intuitively, the interpretation of a formula A is a two-player ($\exists$loise and $\forall$belard) one-move game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ such that the existential player has a winning strategy whenever A is provable in LL^ω. Moreover, the linear logic proof of A will provide a witness $\boldsymbol{a}$ to the fact that $\exists$loise has a winning strategy, i.e. $\forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}$.

Definition 2.1 (Realizability interpretation) *The interpretation of atomic formulas*[2] A_{at} *and* $A^\perp_{\mathsf{at}}$ *are the atomic formulas themselves, i.e.*

$$|A_{\mathsf{at}}| \quad :\equiv \quad A_{\mathsf{at}}$$
$$|A^\perp_{\mathsf{at}}| \quad :\equiv \quad A^\perp_{\mathsf{at}}.$$

Notice that for atomic formulas the tuples of witnesses and challenges are both empty (and hence omitted). We extend that interpretation to all formulas A of LL^ω, as follows. Assume we have already defined $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we define

$$
\begin{array}{lcl}
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{f}} & :\equiv & |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f}z} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|?A|_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!A|^{\boldsymbol{x}} & :\equiv & !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

Note that, given a formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and a sequence of terms $\boldsymbol{a}$ of the same type as the sequence of variables $\boldsymbol{x}$, we write $|A|^{\boldsymbol{a}}_{\boldsymbol{y}}$ for $(|A|^{\boldsymbol{x}}_{\boldsymbol{y}})[\boldsymbol{a}/\boldsymbol{x}]$.

Let us look at a simple example. For instance, consider the valid formula

$$A \equiv !(\forall y A_{\mathsf{at}}(y) \multimap \forall y A_{\mathsf{at}}(y)).$$

Following the description of the interpretation we have:

$$|\forall y A_{\mathsf{at}}(y)|_y \equiv A_{\mathsf{at}}(y).$$

[1] Thanks to Prof. Schwichtenberg for suggesting the name *challenge* rather than *counter-example* variables as used in [15].

[2] Recall that, for simplicity, we omit the treatment of the propositional constants $0, 1, \perp, \top$. They should be treated as the atomic formulas, e.g. $|0| \equiv 0$.

The treatment of the linear implication gives

$$|\forall y A_{\mathsf{at}}(y) \multimap \forall y A_{\mathsf{at}}(y)|^f_y \equiv A_{\mathsf{at}}(fy) \multimap A_{\mathsf{at}}(y).$$

Finally, the *bang* ! makes the challenge variable explicit

$$|A|^f \equiv !\forall y (A_{\mathsf{at}}(fy) \multimap A_{\mathsf{at}}(y)).$$

A proof of A will then provide a *witness* t and a *verification proof* of $|A|^t$. In this simple case, the shortest proof of A will give us the identity term $t = \lambda y.y$.

Following the intuition of games, the interpretation can be understood as follows. The game $A \multimap B$ consists of two simultaneous games A and B, where $\exists$loise must win game B given that $\forall$belard wins game A. More precisely, her move in the game $A \multimap B$ is a pair of copying strategies $(\boldsymbol{f}, \boldsymbol{g})$ transforming any of $\forall$belard's move $\boldsymbol{x}$ in game A into her move $\boldsymbol{g}\boldsymbol{x}$ in game B, and also any of $\forall$belard's move $\boldsymbol{w}$ in game B into her move $\boldsymbol{f}\boldsymbol{w}$ in game A. In the case of the game $A \otimes B$, $\exists$loise must make moves in both games A and B, while $\forall$belard may use $\exists$loise's move in game A to choose his move in game B, and $\exists$loise's move in game B to choose his move in game A.

It is only in the quantifier games $\exists z A(z)$ and $\forall z A(z)$ that the players have to produce a "card" z. In the case of $\exists z A(z)$, $\exists$loise has to produce z and make a move on the game $A(z)$. $\forall$belard's move in the game $A(z)$ can depend on $\exists$loise's choice z. Once again, for the game $\forall z A(z)$ the situation is totally symmetric, so that the onus of producing the witness falls on $\forall$belard.

The exponential games $?A$ and $!A$ can be thought as playing the game A several times. In the case of the game $!A$, $\forall$belard has an advantage since $\exists$loise's move has to be uniform for all copies of the game. $\forall$belard, however, can try a different move on each copy of the game, which is equivalent to $\forall$belard having the chance to play his best possible move to challenge $\exists$loise's move. The roles are reversed in the $?A$ game.

The following lemma, which is an adaptation of a lemma due to Shirahata [19], formalises the idea that the interpretation of $A^\perp$ is identical to that of A save that the roles of players $\exists$loise and $\forall$belard are inverted.

Lemma 2.2 $|A^\perp|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^\perp$.

Definition 2.3 (Fixed formulas) *A formula of LL^ω is called a* fixed formula *if all universal quantifiers are immediately preceded by* !, *and all existential quantifiers are immediately preceded by* ?.

For instance, the formula $!?(\forall x A_{\mathsf{at}}(x) \otimes \exists y B_{\mathsf{at}}(y))$ is not a fixed formula, while $!\forall x (A_{\mathsf{at}}(x) \otimes ?\exists y B_{\mathsf{at}}(y))$ is a fixed formula. Fixed formulas are linear logic counterparts of $\exists$-free formulas which are used in connection with realizability interpretation of intuitionistic logic (cf. [21], remark after 3.4). Such formulas have interesting properties with respect to the realizability interpretation. For instance:

- for any formula A, its interpretation $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is a fixed formula;
- if A is a fixed formula then $|A| \equiv A$ and the sequence of witnessing and challenge variables in its interpretation are empty;
- in particular, we have that the interpretation above is idempotent, i.e. the interpretation of $|A|$ is syntactically equal to $|A|$ itself.

3 Soundness of the Interpretation

We prove now the soundness of the realizability interpretation, i.e. we show that any formula provable in classical linear logic has (provably in classical linear logic) a realizer.

Theorem 3.1 (Soundness) *Let $A_0, \ldots, A_n, B$ be formulas of LL^ω, with $\boldsymbol{z}$ as the only free-variables. If*

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash_{\mathsf{LL}^\omega} B(\boldsymbol{z})$$

then terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n, \boldsymbol{b}$ can be extracted from this proof such that

$$|A_0(\boldsymbol{z})|^{\boldsymbol{x}_0}_{\boldsymbol{a}_0}, \ldots, |A_n(\boldsymbol{z})|^{\boldsymbol{x}_n}_{\boldsymbol{a}_n} \vdash_{\mathsf{LL}^\omega} |B(\boldsymbol{z})|^{\boldsymbol{b}}_{\boldsymbol{y}}$$

where

$$\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\} \backslash \{\boldsymbol{x}_i\}$$

$$\mathsf{FV}(\boldsymbol{b}) \in \{\boldsymbol{z}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}.$$

Proof. The proof is by induction on the derivation of $A_0, \ldots, A_n \vdash B$. The only rule where free-variables matter is the universal quantifier rule. Therefore, for all the other rules we will assume that the tuple of parameters $\boldsymbol{z}$ is empty. The cases of the axiom and the permutation rule are trivial. The other rules are treated as follows:

Tensor.

$$\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x}]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{fb}]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{fb}}}\,[\tfrac{\boldsymbol{fb}}{\boldsymbol{x}}] \quad \dfrac{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{y}]} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{ga}}}\,[\tfrac{\boldsymbol{ga}}{\boldsymbol{y}}]}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{fb}]}, |\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{fb}} \otimes |B|^{\boldsymbol{b}}_{\boldsymbol{ga}}}\,(\otimes)}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{fb}]}, |\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]} \vdash |A \otimes B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{f},\boldsymbol{g}}}\,(\text{D2.1})$$

where $\boldsymbol{f}, \boldsymbol{g}$ are fresh-variables.

Linear implication.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \multimap |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}\,(\multimap)}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash |A \multimap B|^{\lambda \boldsymbol{y}.\boldsymbol{a}, \lambda \boldsymbol{x}.\boldsymbol{b}}_{\boldsymbol{x},\boldsymbol{y}}}\,(\text{D2.1})$$

Contraction.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x}_0,\boldsymbol{x}_1]}, |!A|^{\boldsymbol{x}_0}, |!A|^{\boldsymbol{x}_1} \vdash |B|^{\boldsymbol{b}[\boldsymbol{x}_0,\boldsymbol{x}_1]}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x},\boldsymbol{x}]}, |!A|^{\boldsymbol{x}}, |!A|^{\boldsymbol{x}} \vdash |B|^{\boldsymbol{b}[\boldsymbol{x},\boldsymbol{x}]}_{\boldsymbol{y}}}\,[\tfrac{\boldsymbol{x}}{\boldsymbol{x}_0}, \tfrac{\boldsymbol{x}}{\boldsymbol{x}_1}]}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x},\boldsymbol{x}]}, |!A|^{\boldsymbol{x}} \vdash |B|^{\boldsymbol{b}[\boldsymbol{x},\boldsymbol{x}]}_{\boldsymbol{y}}}\,(\text{con})$$

Weakening.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\text{wkn})}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\text{D2.1})$$

Promotion.

$$\dfrac{\dfrac{\dfrac{|!\Gamma|^{\boldsymbol{v}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|!\Gamma|^{\boldsymbol{v}} \vdash \forall \boldsymbol{x} |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}\,(\forall)}{|!\Gamma|^{\boldsymbol{v}} \vdash !\forall \boldsymbol{x} |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}\,(!)}{|!\Gamma|^{\boldsymbol{v}} \vdash |!A|^{\boldsymbol{a}}}\,(\text{D2.1})$$

Dereliction.

$$\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash \exists \boldsymbol{y} |A|^{\boldsymbol{y}}_{\boldsymbol{x}}}\,(\exists)}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash ?\exists \boldsymbol{y} |A|^{\boldsymbol{y}}_{\boldsymbol{x}}}\,(?)}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}} \vdash |?A|_{\boldsymbol{x}}}\,(\text{D2.1})$$

Cut.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x}]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{a}^-]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{a}^-}}\,[\tfrac{\boldsymbol{a}^-}{\boldsymbol{x}}] \quad \dfrac{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{x}^-]}, |A|^{\boldsymbol{x}^-}_{\boldsymbol{a}^-} \vdash |B|^{\boldsymbol{b}[\boldsymbol{x}^-]}_{\boldsymbol{y}}}{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{a}]}, |A|^{\boldsymbol{a}}_{\boldsymbol{a}^-} \vdash |B|^{\boldsymbol{b}[\boldsymbol{a}]}_{\boldsymbol{y}}}\,[\tfrac{\boldsymbol{a}}{\boldsymbol{x}^-}]}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{a}^-]}, |\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{a}]} \vdash |B|^{\boldsymbol{b}[\boldsymbol{a}]}_{\boldsymbol{y}}}\,(\text{cut})$$

Note that the assumption that the tuple of variables $\boldsymbol{x}$ (respectively $\boldsymbol{x}^-$) does not appear free in the term $\boldsymbol{a}$ (respectively $\boldsymbol{a}^-$) is used crucially in the soundness of the cut rule in order remove any circularity in the two simultaneous substitutions.

Universal quantifier.

$$\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[z]} \vdash |A(z)|^{\boldsymbol{a}[z]}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[z]} \vdash |\forall z A(z)|^{\lambda z.\boldsymbol{a}[z]}_{\boldsymbol{x},z}}\,(\text{D2.1})$$

Existential quantifier.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x}]} \vdash |A(t)|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{g}t]} \vdash |A(t)|^{\boldsymbol{a}}_{\boldsymbol{g}t}}\,[\tfrac{\boldsymbol{g}t}{\boldsymbol{x}}]}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{g}t]} \vdash |\exists z A(z)|^{\boldsymbol{a},t}_{\boldsymbol{g}}}\,(\text{D2.1})$$

Linear swap.

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, (|B|^{\boldsymbol{b}}_{\boldsymbol{y}})^\perp \vdash (|A|^{\boldsymbol{x}}_{\boldsymbol{a}})^\perp}\,(\perp)}{|\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}}, |B^\perp|^{\boldsymbol{y}}_{\boldsymbol{b}} \vdash |A^\perp|^{\boldsymbol{a}}_{\boldsymbol{x}}}\,(\text{L2.2})$$

This concludes the proof. $\square$

As an immediate consequence of Theorem 3.1 we get the following:

Corollary 3.2 *The system LL^ω has the existence property for fixed formulas. More precisely, if $A_0, \ldots, A_n, B$ are fixed formulas and*

$$A_0, \ldots, A_n \vdash_{\mathsf{LL}^\omega} \forall \boldsymbol{x} \exists \boldsymbol{y} B(\boldsymbol{x}, \boldsymbol{y})$$

then $A_0, \ldots, A_n \vdash_{\mathsf{LL}^\omega} \forall \boldsymbol{x} B(\boldsymbol{x}, \boldsymbol{t}\boldsymbol{x})$, for some sequence of terms $\boldsymbol{t}$.

We discuss the case of the disjunction property in Section 5, where we treat the additive connectives.

4 Completeness of the Interpretation

In this section we investigate the completeness of the realizability interpretation. The Soundness Theorem 3.1 tells us that whenever $\Gamma \vdash A$ then for terms $\boldsymbol{a}[\boldsymbol{v}], \boldsymbol{\gamma}[\boldsymbol{y}]$ we also have

$$(*) \quad |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{y}]} \vdash |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{y}},$$

treating, for simplicity, the context Γ as a single formula. Our goal is to define a minimal extension of LL^ω over which having $(*)$ also implies that $\Gamma \vdash A$. Since the interpretation of A will normally have more free-variables than the formula A, we must choose how to quantify over these variables. Given that $\boldsymbol{a}$ is independent of $\boldsymbol{y}$, and $\boldsymbol{\gamma}$ is independent of $\boldsymbol{v}$, one might be tempted to say that a sequent $\Gamma \vdash A$ is interpreted as

$$(1) \quad \forall \boldsymbol{w} \exists \boldsymbol{v} |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

As discussed in [4], interpretation (1) is not in general sound for the logical axioms. Given the rule $(\perp)$, formulas in the premise of the sequent must be given a dual interpretation to the formula in the conclusion. Therefore, one might try also to consider the alternative possibility

$$(2) \quad \exists \boldsymbol{v} \forall \boldsymbol{w} |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash \forall \boldsymbol{y} \exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Interpretation (2) validates the axioms but will not validate the cut rule, since we will need $\forall \boldsymbol{y} \exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. In order to obtain an interpretation between options (1) and (2), we use a simple form of branching quantification, to be called *simultaneous quantifiers* $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$, where $\boldsymbol{x}, \boldsymbol{y}$ are tuples of variables. Using the simultaneous quantifiers, we can view the sequent $\Gamma \vdash A$ as

$$(3) \quad \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

With the sequent $(*)$ in mind, the simultaneous quantifier aims at capturing both the global dependence ($\boldsymbol{a}$ depends on $\boldsymbol{v}$) and local independence ($\boldsymbol{a}$ does not depend on $\boldsymbol{y}$) of the witnessing terms.

The logical rule for this simple form of branching quantifier is:

$$\frac{A_0(\boldsymbol{x}_0, \boldsymbol{a}_0), \ldots, A_n(\boldsymbol{x}_n, \boldsymbol{a}_n) \vdash B(\boldsymbol{b}, \boldsymbol{w})}{\exists\!\!\!\forall^{\boldsymbol{x}_0}_{\boldsymbol{y}_0} A_0(\boldsymbol{x}_0, \boldsymbol{y}_0), \ldots, \exists\!\!\!\forall^{\boldsymbol{x}_n}_{\boldsymbol{y}_n} A_n(\boldsymbol{x}_n, \boldsymbol{y}_n) \vdash \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B(\boldsymbol{v}, \boldsymbol{w})} \; (\exists\!\!\!\forall)$$

with the two side-conditions:

- $\boldsymbol{x}_i$ may only appear free in the terms $\boldsymbol{b}$ or $\boldsymbol{a}_j$, for $j \neq i$;
- $\boldsymbol{w}$ may only appear free in the terms $\boldsymbol{a}_i$.

In particular, we will have that $\boldsymbol{w}$ and each $\boldsymbol{x}_i$ will not be free in the conclusion of the rule. Note that we might have $\boldsymbol{x}_i \in \mathsf{FV}(\boldsymbol{a}_j) \cup \mathsf{FV}(\boldsymbol{b})$, for $j \neq i$, and $\boldsymbol{w} \in \mathsf{FV}(\boldsymbol{a}_i)$.

The standard quantifier rules can be obtained from this single rule. The rule $(\forall)$ can be obtained in the case when $\boldsymbol{x}_i, \boldsymbol{a}_i$ and $\boldsymbol{b}$ are empty. The rule $(\exists)$ can be obtained in the case when $\boldsymbol{x}_i, \boldsymbol{a}_i$ and $\boldsymbol{w}$ are empty. Hence, for the rest of this section we will consider that standard quantifiers $\forall x A$ and $\exists x A$ are in fact abbreviations for $\exists\!\!\!\forall_x A$ and $\exists\!\!\!\forall^x A$, respectively.

Remark 4.1 (Relation to Henkin quantifiers) *According to Hyland [13] (footnote 18), "the identification of a sufficiently simple tensor as a Henkin quantifier is a common feature of a number of interpretations of linear logic". The simultaneous quantifier can be viewed as a simplification of Henkin's (branching) quantifier [5, 12], in which no alternation of quantifiers is allowed on the two branches.*

In terms of games, the simple branching quantifier embodies the idea of the two players performing their moves simultaneously. The most interesting characteristic of this simultaneous quantifier is with respect to linear negation, which is defined as

$$(\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A)^\perp \equiv \exists\!\!\!\forall^{\boldsymbol{y}}_{\boldsymbol{x}} A^\perp$$

and corresponds precisely to the switch of roles between the players. Let us refer to the extension of LL^ω with the simultaneous quantifier by $\mathsf{LL}^\omega_{\mathsf{q}}$.

We can also extend the realizability interpretation (Definition 2.1) to the system $\mathsf{LL}^\omega_{\mathsf{q}}$ as follows. If $A(\boldsymbol{v}, \boldsymbol{w})$ has interpretation $|A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{x}}_{\boldsymbol{y}}$ then

$$|\exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}, \boldsymbol{v}}_{\boldsymbol{g}, \boldsymbol{w}} \quad :\equiv \quad |A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{g}\boldsymbol{v}}.$$

Lemma 4.2 *Given this interpretation of the simultaneous quantifier, Theorem 3.1 can be extended to the system $\mathsf{LL}^\omega_{\mathsf{q}}$.*

In fact, since the simultaneous quantifiers are eliminated, we obtain an interpretation of $\mathsf{LL}^\omega_{\mathsf{q}}$ into LL^ω.

Let us proceed now to define an extension of $\mathsf{LL}^\omega_{\mathsf{q}}$ which is complete with respect to the modified realizability interpretation. First we need some simple facts about $\mathsf{LL}^\omega_{\mathsf{q}}$.

Lemma 4.3 *The following are derivable in* $\mathsf{LL}^{\omega}_{\mathsf{q}}$

(i) $\exists\!\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y},z} A(\boldsymbol{f}z, \boldsymbol{y}, z) \multimap \forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z)$

(ii) $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{y}) \otimes \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B(\boldsymbol{w}) \multimap \exists\!\!\!\forall^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} (A(\boldsymbol{f}\boldsymbol{v}) \otimes B(\boldsymbol{g}\boldsymbol{x}))$

(iii) $\exists \boldsymbol{x} ! \forall \boldsymbol{y} A \multimap ! \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$.

The converses of these implications, however, require extra logical principles. Let A and B be *fixed formulas* (cf. Definition 2.3), and consider the following principles for the simultaneous quantifier[3]

(AC_s) $\forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z) \multimap \exists\!\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y},z} A(\boldsymbol{f}z, \boldsymbol{y}, z)$

(AC_p) $\exists\!\!\!\forall^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} (A(\boldsymbol{f}\boldsymbol{v}) \otimes B(\boldsymbol{g}\boldsymbol{x})) \multimap \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{y}) \otimes \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B(\boldsymbol{w})$

(TA) $! \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \multimap \exists \boldsymbol{x} ! \forall \boldsymbol{y} A$.

We refer to these as the *sequential choice* AC_s, *parallel choice* AC_p, and *trump advantage* TA.

Lemma 4.4 *The principles* AC_s, AC_p *and* TA *are sound for the realizability interpretation, i.e. for any instance P of these principles, there are terms $\boldsymbol{t}$ such that* $\vdash_{\mathsf{LL}^{\omega}} |P|^{\boldsymbol{t}}_{\boldsymbol{y}}$.

Let us denote by $\mathsf{LL}^{\omega}_{\mathsf{q}+}$ the extension of $\mathsf{LL}^{\omega}_{\mathsf{q}}$ with these three extra schemata. Lemma 4.4 implies:

Lemma 4.5 *If* $\vdash_{\mathsf{LL}^{\omega}_{\mathsf{q}+}} A$ *then* $\vdash_{\mathsf{LL}^{\omega}_{\mathsf{q}}} \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

The next lemma shows that, in fact, these extra principles are all one needs to show the equivalence between A and its interpretation $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

Lemma 4.6 $\vdash_{\mathsf{LL}^{\omega}_{\mathsf{q}+}} A \leftrightarrow \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

Theorem 4.7 *Let A be a formula in the language of* LL^{ω}. *Then* $\vdash_{\mathsf{LL}^{\omega}_{\mathsf{q}+}} A$ *if and only if* $\vdash_{\mathsf{LL}^{\omega}} |A|^{\boldsymbol{t}}_{\boldsymbol{y}}$, *for some term* $\boldsymbol{t}$.

Proof. The forward direction follows from the Soundness Theorem 3.1 and Lemma 4.4. The converse follows from Lemma 4.6. $\Box$

Corollary 4.8 $\mathsf{LL}^{\omega}_{\mathsf{q}+}$ *is conservative over* LL^{ω} *for fixed formulas.*

[3]For those familiar with the modified realizability of intuitionistic logic, the principle AC_s corresponds to the standard axiom of choice, while AC_p is a generalisation of the independence of premise principle (case when tuples $\boldsymbol{x}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are empty).

$$\frac{\Gamma[\gamma_0] \vdash A \quad \Gamma[\gamma_1] \vdash B}{\Gamma[(z)(\gamma_0, \gamma_1)] \vdash A \diamond_z B} \,(\diamond_z)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \diamond_{\mathrm{t}} B} \,(\diamond_{\mathrm{t}}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \diamond_{\mathrm{f}} B} \,(\diamond_{\mathrm{f}})$$

Table 2. Rules for if-then-else connective

5 Dealing with the Additives

In this section we discuss how the realizability interpretation of Section 2 can be extended to deal with the additive connectives. We will deviate from the standard formulation of linear logic, in the sense that we will use the if-then-else logical constructor $A \diamond_z B$ instead of standard additive conjunction and disjunction[4]. The logical rules for $\diamond_z$ are shown in Table 2, where $(z)(\gamma_0, \gamma_1)$ denotes a conditional λ-term which reduces to either γ_0 or γ_1 depending on whether the boolean variable z reduces to true or false, respectively. The standard additives can be defined as

$$A \wedge B :\equiv \forall z (A \diamond_z B)$$

$$A \vee B :\equiv \exists z (A \diamond_z B)$$

with the help of quantification over booleans[5]. The if-then-else connective is self-dual, i.e.

$$(A \diamond_z B)^{\perp} \equiv A^{\perp} \diamond_z B^{\perp}.$$

The realizability interpretation given in Definition 2.1 can be extended to deal with the if-then-else construct as

$$|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}.$$

Therefore, the treatment of conjunction and disjunction can be given directly as

$$\begin{array}{lcl} |A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} & \equiv & |A|^{\boldsymbol{x}z}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}z}_{\boldsymbol{w}} \\ |A \vee B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} & \equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}z} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}z}. \end{array}$$

Moreover, notice that the functionals $\boldsymbol{x}$ and $\boldsymbol{v}$ in the case of $\wedge$ ($\boldsymbol{y}$ and $\boldsymbol{w}$ in the case of $\vee$) do not need to have access to the boolean z, since they will only be relevant when $z = \mathrm{t}$ and $z = \mathrm{f}$, respectively. The interpretation can then be simplified as

[4]See Girard's comments in [9] (p13) and [10] (p73) on the relation between the additive connectives and the if-then-else construct.

[5]As discussed in Remark 1.1, these extra apparatuses are only needed in the verifying system, so that the interpretation can still be seen as interpreting full classical linear logic LL into the (less standard) system we are describing here.

$$
\begin{array}{lcl}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} & \equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \vee B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} & \equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}.
\end{array}
$$

In terms of games, given two games A and B, the additive combinations of those games, i.e. $A \wedge B$ and $A \vee B$, correspond to two games being played simultaneously, but only the outcome of one of those games will count at the end. The game whose outcome will count depends on whether it is an *and* ($\wedge$) game or an *or* ($\vee$) game. In the first case, $\forall$belard must choose not only two moves $\boldsymbol{y}, \boldsymbol{w}$, but he chooses also a boolean deciding which game he wants to count. In the case of the or-game, $\exists$loise has the choice of which games is to be considered.

The Soundness Theorem 3.1 remains true for the extension with the if-then-else connective. For instance, the conditional abstraction is treated as follows:

$$
\dfrac{\dfrac{|\Gamma|^{\boldsymbol{v}}_{\gamma_0} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}} \qquad |\Gamma|^{\boldsymbol{v}}_{\gamma_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{v}}_{(z)(\gamma_0,\gamma_1)} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}} \diamond_z |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}\,(\diamond_z)}{|\Gamma|^{\boldsymbol{v}}_{(z)(\gamma_0,\gamma_1)} \vdash |A \diamond_z B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{x},\boldsymbol{y}}}\,(\text{D2.1})
$$

Using the abbreviation $B \vee C :\equiv \exists z(B \diamond_z C)$, Corollary 3.2 implies the following disjunction property.

Corollary 5.1 *The system LL^ω has the disjunction property for fixed formulas. More precisely, if $A_0, \ldots, A_n, B, C$ are closed fixed formulas and*

$$A_0, \ldots, A_n \vdash_{\mathsf{LL}^\omega} B \vee C$$

then, either $A_0, \ldots, A_n \vdash_{\mathsf{LL}^\omega} B$ or $A_0, \ldots, A_n \vdash_{\mathsf{LL}^\omega} C$.

6 Relation to Kreisel's Modified Realizability

In this section we describe how the interpretation of linear logic presented above indeed corresponds to Kreisel's modified realizability interpretation [14] of intuitionistic logic. We will assume that intuitionistic logic is also formalised with the if-then-else connective $A \diamond_b B$, so that conjunction and disjunction are defined notions.

First, consider a variation of Girard's embedding of intuitionistic logic into our version of linear logic with conditionals.

Definition 6.1 ([9]) *For any formula A of intuitionistic logic its linear translation $A^\star$ is defined inductively as*

$$
\begin{array}{lcl}
A^\star_{\mathsf{at}} & :\equiv & A_{\mathsf{at}} \\
(A \diamond_b B)^\star & :\equiv & A^\star \diamond_b B^\star \\
(A \to B)^\star & :\equiv & !A^\star \multimap B^\star \\
(\forall x A)^\star & :\equiv & \forall x A^\star \\
(\exists x A)^\star & :\equiv & \exists x !A^\star.
\end{array}
$$

$$
\begin{tikzcd}
\mathsf{IL}^\omega \arrow[r, "\mathsf{mr}"] \arrow[d, "!(\cdot)^\star"'] & \mathsf{IL}^\omega \arrow[d, "!(\cdot)^\star"] \\
\mathsf{LL}^\omega \arrow[r, "|\cdot|"] & \mathsf{LL}^\omega
\end{tikzcd}
$$

Figure 1. Interpreting IL^ω into LL^ω

The translation is such that $\Gamma \vdash_{\mathsf{IL}^\omega} A$ if and only if $!(\Gamma^\star) \vdash_{\mathsf{LL}^\omega} A^\star$, for all formulas Γ, A in the language of intuitionistic logic.

Let "$\boldsymbol{x}\ \mathsf{mr}\ A$" denote the modified realizability interpretation of A, read as "$\boldsymbol{x}$ modified realizes A". The next theorem basically states that the diagram of Figure 1 commutes.

Theorem 6.2 *Let A be a formula of intuitionistic logic. Then $\vdash_{\mathsf{LL}^\omega} !(\boldsymbol{x}\ \mathsf{mr}\ A)^\star \leftrightarrow |!A^\star|^{\boldsymbol{x}}$.*

It is easy to check that the embedding of a formula A of classical logic into linear logic via the translation:

$$
\begin{array}{lcl}
A^Q_{\mathsf{at}} & :\equiv & A_{\mathsf{at}} \\
(A \to B)^Q & :\equiv & !A^Q \multimap ?!B^Q \\
(\forall x A)^Q & :\equiv & \forall x ?!A^Q
\end{array}
$$

is equivalent to a $\forall$-fixed formula, i.e. $\forall x A$, with A a fixed formula. This implies that the realizability interpretation of A^Q does not have any positive computational content, and that the universal closure of the realizability interpretation of A^Q is equivalent to A^Q itself. Intuitively, Girard's embedding of classical logic into linear logic "blocks" the realizability interpretation (Def. 2.1) in the same way that the negative translation "blocks" Kreisel's original realizability interpretation.

7 Parametrised Interpretation and LL^ω

In [15], a family of functional interpretations of intuitionistic logic has been described using a unifying framework. Such framework can be translated to the case of classical linear logic as follows. For each formula A, let $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ be formula abbreviations such that

$$
\begin{array}{lcl}
(\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A)^\perp & \equiv & \exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A^\perp \\
(\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A)^\perp & \equiv & \forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A^\perp.
\end{array}
$$

A formula A is called $\sqsubset$-fixed if it does not contain unbounded quantifiers and all bounded quantifiers $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ are immediately preceded by a ! and ?, respectively. For each $\sqsubset$-fixed formula A, we require the abbreviations to satisfy the *comonoid condition*

(C) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\epsilon} \boldsymbol{y}_0 \boldsymbol{y}_1 \, A(\boldsymbol{y}) \vdash \, !\forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i \, A(\boldsymbol{y}) \qquad (i \in \{0,1\})$

and the *comonad conditions*

(D) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\eta x} \, A(\boldsymbol{y}) \vdash A(\boldsymbol{x})$

(P) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{\mu h w} \, A(\boldsymbol{y}) \vdash \, !\forall \boldsymbol{x} \sqsubset \boldsymbol{w} \; !\forall \boldsymbol{y} \sqsubset \boldsymbol{hx} \, A(\boldsymbol{y})$

for families of sequence of terms $\boldsymbol{\epsilon}, \boldsymbol{\eta}$ and $\boldsymbol{\mu}$, one sequence of terms for each $\sqsubset$-fixed formula A. The provability sign in the conditions stands for provability in the system under which the functional interpretation will be verified, which might be an extension of LL^ω. For a sequence of formulas $\Gamma \equiv A_0, \ldots, A_n$ we will write $\forall \boldsymbol{x} \sqsubset \boldsymbol{a} \, \Gamma$ as a shorthand for $\forall \boldsymbol{x}_0 \sqsubset \boldsymbol{a}_0 \, A_0, \ldots, \forall \boldsymbol{x}_n \sqsubset \boldsymbol{a}_n \, A_n$.

We can then show that, for any formula abbreviation satisfying conditions (C, D, P), a functional interpretation of classical linear logic can be obtained by defining the interpretation of the exponentials as

$$
\begin{array}{lll}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !\forall \boldsymbol{y} \sqsubset \boldsymbol{fx} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} \sqsubset \boldsymbol{fy} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

The conditions (D, P) are used to ensure the soundness of the dereliction and promotion rules, respectively. For instance, the parametrised soundness of the promotion rule is derivable as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{|!\Gamma|^{\boldsymbol{v}}_{\boldsymbol{\gamma}[\boldsymbol{x}]} \vdash |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{x}}}{!\forall \boldsymbol{w} \sqsubset \boldsymbol{hx} \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{x}}}\,(\mathrm{D}2.1)}{!\forall \boldsymbol{x} \sqsubset \boldsymbol{b} \, !\forall \boldsymbol{w} \sqsubset \boldsymbol{hx} \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash \, !\forall \boldsymbol{x} \sqsubset \boldsymbol{b} \, |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{x}}}\,(+)}{!\forall \boldsymbol{w} \sqsubset \boldsymbol{\mu h b} \, |\Gamma|^{\boldsymbol{v}}_{\boldsymbol{w}} \vdash \, !\forall \boldsymbol{x} \sqsubset \boldsymbol{b} \, |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{x}}}\,(\mathsf{P})}{|!\Gamma|^{\boldsymbol{v}}_{\lambda \boldsymbol{v}.\boldsymbol{\mu h b}} \vdash |!A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{f}}}\,(\mathrm{D}2.1)
$$

where $\boldsymbol{h} \equiv \lambda \boldsymbol{x}.\boldsymbol{\gamma}[\boldsymbol{x}]\boldsymbol{v}$ and $\boldsymbol{b} \equiv \boldsymbol{f}(\boldsymbol{a}[\boldsymbol{v}])$. The step (+) in the derivation above corresponds to several steps in linear logic, assuming also that the abbreviation respects provability. The condition (C) is used for the soundness of the contraction rule.

Modified realizability is the case when the tuple $\boldsymbol{a}$ is empty and the formula abbreviations $\forall \boldsymbol{x} \sqsubset \boldsymbol{a} \, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a} \, A$ are shorthand for $\forall \boldsymbol{x} A$ and $\exists \boldsymbol{x} A$, respectively. It is easy to see that with this abbreviation, all the three conditions are satisfied for LL^ω.

In the following, we briefly discuss three other instantiations of the parametrised interpretation, corresponding the Gödel's Dialectica interpretation [11], its Diller-Nahm variant [6], and the bounded functional interpretation [7]. For each variant we will focus on the requirements for the soundness theorem to go through. For lack of space we omit the proofs that these interpretations of classical linear logic indeed correspond (via the $(\cdot)^\star$-translation) to their respective interpretations of intuitionistic logic. The proof, however, is similar to the proof of Theorem 6.2.

As mentioned above, the parametrised formula construction is only necessary in the treatment of exponentials. Hence, in a fragment of linear logic without exponentials, all the functional interpretations we will discuss below coincide. This also means that the completeness proof of Section 4 can be reused, and only new principles for the completeness of the exponentials needs to be added. Using parametrised notation, the only new principle we need for the (parametrised) completeness of the exponentials is the linear logic counterpart of the Markov principle

($\mathsf{MP}_\sqsubset$) $\forall \boldsymbol{z} ! \forall \boldsymbol{x} \sqsubset \boldsymbol{z} \, A \multimap \, !\forall \boldsymbol{x} A$

where A is a $\sqsubset$-fixed formula. With $\mathsf{MP}_\sqsubset$ we can obtain the equivalence between $!\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$ and $\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{f}} ! \forall \boldsymbol{y} \sqsubset \boldsymbol{fx} \, A$ as:

$$
\begin{tikzcd}
!\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \arrow[r, shift left, "\mathsf{TA}"] & \exists \boldsymbol{x} ! \forall \boldsymbol{y} A \arrow[l, shift left, "\mathsf{LL}^\omega_{\mathsf{q}}"] \arrow[r, shift left, "\mathsf{LL}^\omega"] & \exists \boldsymbol{x} \forall \boldsymbol{z} ! \forall \boldsymbol{y} \sqsubset \boldsymbol{z} \, A \arrow[l, shift left, "\mathsf{MP}_\sqsubset"] \arrow[r, shift left, "\mathsf{LL}^\omega_{\mathsf{q}}"] & \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{f}} ! \forall \boldsymbol{y} \sqsubset \boldsymbol{fx} \, A \arrow[l, shift left, "\mathsf{AC}_s"]
\end{tikzcd}
$$

In the case of the modified realizability interpretation, $\mathsf{MP}_\sqsubset$ reduces to the trivial implication $!\forall \boldsymbol{x} A \multimap \, !\forall \boldsymbol{x} A$.

7.1 Gödel's Dialectica interpretation

In [11], Gödel developed an interpretation of Heyting (first-order intuitionistic) arithmetic into a higher-order extension of primitive recursive arithmetic. Gödels' interpretation of arithmetic, known as the *Dialectica interpretation*, is based on top of an interesting interpretation of intuitionistic predicate logic. This interpretation has been adapted to an interpretation of propositional linear logic in [16, 17], and later extended to first-order linear logic in [19]. Gödel's Dialectica interpretation of linear logic can be obtained from the parametrised interpretation by taking the abbreviations $\forall \boldsymbol{x} \sqsubset \boldsymbol{a} \, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a} \, A$ to both mean $A[\boldsymbol{a}/\boldsymbol{x}]$. This leads to a variant of the interpretation given in Def. 2.1 with the interpretation of the exponentials as

$$
\begin{array}{lll}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !|A|^{\boldsymbol{x}}_{\boldsymbol{fx}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?|A|^{\boldsymbol{fy}}_{\boldsymbol{y}}.
\end{array}
$$

In this case, we must assume that quantifier-free formulas are decidable (a usual requirement for Dialectica interpretations) in order to satisfy the condition (C). Conditions (D, P) are easily seen to be satisfied in LL^ω, with $\boldsymbol{\eta x} := \boldsymbol{x}$ and $\boldsymbol{\mu h b} := \boldsymbol{hb}$.

Besides being sound for the principles AC_s, AC_p and TA of Section 4, the Dialectica interpretation of LL^ω will also interpret the corresponding instantiation of $\mathsf{MP}_\sqsubset$, i.e.

(MP_D) $\forall \boldsymbol{x} ! A \multimap \, !\forall \boldsymbol{x} A$

where A is a quantifier-free formula.

7.2 Diller-Nahm interpretation

A variant of Gödel's Dialectica interpretation which does not require decidability of quantifier-free formulas is the Diller-Nahm interpretation [6]. In fact, de Paiva's Dialectica interpretation of linear logic [16, 17] is closer to Diller-Nahm's variant than to Gödels' original interpretation. The Diller-Nahm interpretation of linear logic is also discussed in [19]. The idea is to work with finite sets, so that no particular choices need to be made, but candidate witnesses are simply collected postponing the decisions indefinitely. The Diller-Nahm variant can be obtained from the parametrised interpretation by taking the abbreviations $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ to mean $\forall \boldsymbol{x} \in \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \in \boldsymbol{a}\, A$, respectively, where $\boldsymbol{a}$ is a sequence of finite multi-sets (or sequence of finite sequences). This leads to the following interpretation of the exponentials:

$$
\begin{aligned}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} &:\equiv\ !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} &:\equiv\ ?\exists \boldsymbol{x} \in \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{aligned}
$$

Intuitively, rather than producing a precise witness (or challenge) $\boldsymbol{f}\boldsymbol{x}$, as in Section 7.1, we only need to produce a set of possible witnesses (or challenges). It is clear that we must add enough term construction to the language in order to be able to manipulate these finite sets. Conditions (C, D, P) are seen to be satisfied if we take $\boldsymbol{\epsilon}\boldsymbol{y}_0\boldsymbol{y}_1 := \boldsymbol{y}_0 \cup \boldsymbol{y}_1$, $\boldsymbol{\eta}\boldsymbol{x} := \{\boldsymbol{x}\}$, and $\boldsymbol{\mu}\boldsymbol{h}\boldsymbol{b} := \bigcup_{\boldsymbol{x}\in\boldsymbol{b}}(\boldsymbol{h}\boldsymbol{x})$ using basic manipulations of finite sets.

7.3 Bounded functional interpretation

In [7, 8], a "bounded" variant of Gödel's Dialectica interpretation was developed in order to deal with strong analytical principles in classical feasible analysis. The interpretation makes use of Howard-Bezem's majorizabilty relation $\leq^*$ between functionals (cf. [3]). As shown in [15], also the bounded functional interpretation fits nicely into the unifying framework. In the setting of classical linear logic, the bounded functional interpretation corresponds to the formula abbreviations $\forall \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \sqsubset \boldsymbol{a}\, A$ denoting $\forall \boldsymbol{x} \leq^* \boldsymbol{a}\, A$ and $\exists \boldsymbol{x} \leq^* \boldsymbol{a}\, A$, respectively, which leads to the following interpretation of the exponentials:

$$
\begin{aligned}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} &:\equiv\ !\forall \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} &:\equiv\ ?\exists \boldsymbol{x} \leq^* \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{aligned}
$$

As argued in [15], in this case we must first perform a relativisation of the quantifiers to Bezem's model $\mathcal{M}$ of strongly majorizable functionals. The importance of the relativisation is to cope with condition (D), which is not valid unless $\boldsymbol{x}$ is self-majorizing. The corresponding Markov principle in this case is

$$(\mathsf{MP}_B)\ \ \forall \boldsymbol{b} !\forall \boldsymbol{x} \leq^* \boldsymbol{b}\, A \multimap\ !\forall \boldsymbol{x} A$$

or equivalently $?\exists \boldsymbol{x} A \multimap \exists \boldsymbol{b} ?\exists \boldsymbol{x} \leq^* \boldsymbol{b}\, A$, which together with the principles $\mathsf{AC}_s, \mathsf{AC}_p, \mathsf{TA}$ can be used to derive (linear logic versions of) the extra principles used in the bounded functional interpretation [7].

8 Conclusions and Related Work

As mentioned in the introduction, the basis of the modified realizability interpretation presented here comes from de Paiva's work [16, 17], recently extended by Shirahata [19]. The only difference to our modified realizability interpretation is in the treatment of the exponentials, which are given a much simpler treatment, following a suggestion of Blass [4].

Although the modified realizability interpretation has been presented as a syntactic proof interpretation of LL^ω into the fragment of fixed formulas $\mathsf{LL}^\omega_{\mathsf{fix}}$, it can also be understood semantically via a subsequent interpretation of $\mathsf{LL}^\omega_{\mathsf{fix}}$ into classical logic:

A	$\mapsto$	A^{c}
$\Diamond_z$		$\Diamond_z$
$\multimap$		$\to$
$\otimes$		$\wedge$
$!\forall \boldsymbol{x}$		$\forall \boldsymbol{x}$
$?\exists \boldsymbol{x}$		$\exists \boldsymbol{x}$

It is easy to show that if a formula A in the fixed fragment of LL^ω is derivable then A^{c} is classically true. This implies that if $\vdash_{\mathsf{LL}^\omega} A$, for arbitrary A, then, for some sequence of terms $\boldsymbol{t}$, the formula $(|A|^{\boldsymbol{t}}_{\boldsymbol{y}})^{\mathsf{c}}$ is classically true. By erasing the exponentials, however, the combined interpretation into classical logic identifies $?A$ with A, and hence (as noticed by Shirahata [19]), the combined interpretation validates the standard weakening rule as opposed to the more restricted weakening rule of linear logic. This leads to a combined interpretation of affine logic, rather than pure linear logic.

We can also read the modified realizability interpretation as associating formulas A of linear logic with Chu spaces $(\tau_{\boldsymbol{x}}, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}, \tau_{\boldsymbol{y}})$, where $\tau_{\boldsymbol{x}}$ and $\tau_{\boldsymbol{y}}$ are the types of the tuples of variables $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively (see [2] for a survey on Chu spaces). Therefore, the treatment of each of logical constructors of linear logic corresponds to a construction of new Chu spaces from previous ones. Given a singleton set $\{*\}$, corresponding to the type of the empty tuples, the modified realizability treatment of the exponentials corresponds to the construction of a Chu space $(\{*\}, ?r, Y)$ out of a given Chu space (X, r, Y), where the relation $?r$ is defined as $?r(*, y) :\equiv \exists x \in X\, r(x, y)$. For more details on the relation between the functional interpretations of linear logic and Chu spaces see [4, 18].

The relation between functional interpretations and games is discussed in [4] (see also [1]). As mentioned above, our modified realizability interpretation of the exponentials corresponds to a maximal advantage for each of the players, in the sense that the players are not required to make a move, but will win if their best possible move is a winning move. For an analysis of the different functional interpretations from a categorical-logic point of view see [13], where linear logic is also discussed.

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