Rates of mixing and ergodic averages.

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1. Ergodicity and mixing

Let $\begin{cases} T : (X,\mu) \to (X,\mu) \text{ be a measure} \\ \text{preserving transformation } (\mu(X)=1) \end{cases}$

Ergodicity For $f \in L^2(X,\mu)$, $\int f d\mu = 0$ then

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to 0 \quad \text{as } N \to \infty$$

for a.e. (μ) $x \in X$

[Birkhoff Ergodic Theorem]

(Strong) Mixing For $f \in L^2(X,\mu)$, $\int f d\mu = 0$ then

$$\rho_f(n) := \int f \circ T^n \cdot f d\mu \to 0 \text{ as } n \to \infty.$$

Remark: Mixing $\Longrightarrow$ Ergodic

Mixing $\not\Longleftarrow$ Ergodic

Question: How are the speeds of convergence related in these two definitions?

Theorem (Gaposhkin, cf. Kacharovski; RMS '96)

If $\rho_f(n) = O(n^{-\alpha})$

then $$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \begin{cases} O(N^{-\alpha/2+\varepsilon}) & \text{if } 0<\alpha<1 \\ O(N^{-1/2+\varepsilon}) & \text{if } \alpha \geq 1 \end{cases}$$

for a.e. (μ) $x \in X$ (and any $\varepsilon > 0$).

Remark If $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = O(1/N)$ for a.e. (μ) $x \in X$ then $f = h\circ T - h$ for some $h \in L^\infty(X,\mu)$

(Halasz)

2. Easy examples

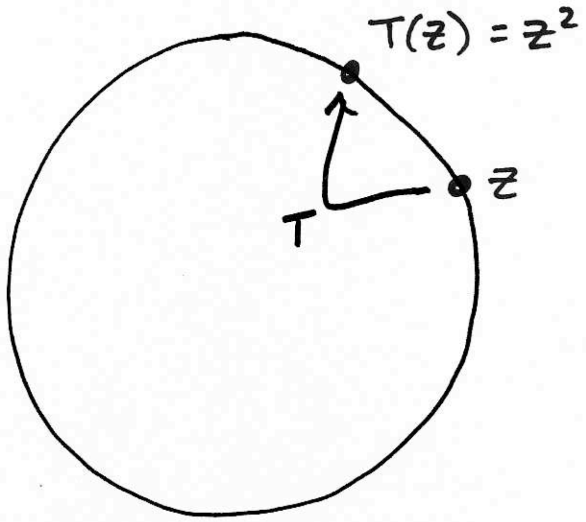
Example 1: Let $\underbrace{\mathbb{T}^1 = \{z \in \mathbb{C} \mid |z| = 1\}}_{\text{unit circle}}$

Let $\begin{cases} T : \mathbb{T}^1 \longrightarrow \mathbb{T}^1 \\ T : z \longmapsto z^2 \end{cases}$

μ = Haar measure

Proposition If $f \in C^\infty(\mathbb{T}^1)$ then for any $\alpha > 0$: $\rho_f(n) = O(e^{-\alpha n})$

(The proof just uses Fourier series)



<u>Remark</u>. The similar looking transformation

$$\begin{cases} S : [0,1) \longrightarrow [0,1) \\ S : x \longmapsto 2x \pmod 1), \end{cases}$$

with μ = Lebesgue measure, has different properties: For $f \in C^\infty([0,1])$, $\int f \, d\mu = 0$ then $\rho_f(n) = O(1/2^n)$

<u>Problem</u> Given a fixed $\alpha > 0$ can we find an expanding map $T : [0,1) \to [0,1)$ of degree 2 which preserves Lebesgue measure μ such that: For $f \in C^\infty([0,1])$, $\int f \, d\mu = 0$ then $\rho_f(n) = O(e^{-\alpha n})$.

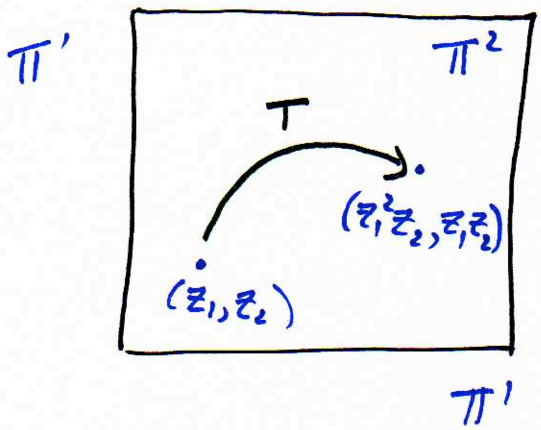
Example 2: Let $\mathbb{T}^2 = \underbrace{\mathbb{T}^1 \times \mathbb{T}^1}_{\text{2 dimensional torus.}}$

Let $\begin{cases} T : \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \\ T : (z_1, z_2) \longmapsto (z_1^2 z_2, z_1 z_2) \end{cases}$

μ = Haar measure on $\mathbb{T}^2$.

Proposition If $f \in C^\infty(\mathbb{T}^2)$ then for any $\alpha > 0$: $\rho_f(n) = O(e^{-\alpha n})$.

(The proof just uses Fourier series)

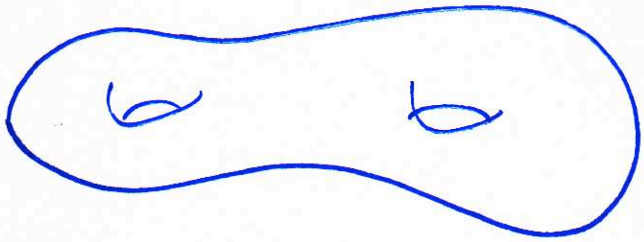


3. Geodesic flows

Let $\begin{cases} G = SL(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{matrix} \right\} \\ \Gamma < G \text{ be a discrete subgroup} \\ \quad \text{(such that } G/\Gamma \text{ is compact)} \\ g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \text{ for } t \in \mathbb{R} \\ M = \{\pm I\} \backslash G/\Gamma \end{cases}$

Let $\begin{cases} \phi_t : M \to M \\ \phi_t(\{\pm I\} g\Gamma) = \{\pm I\} g_t g \Gamma \end{cases}$ "geodesic flow"

Remark: Corresponds to the geodesic flow on the unit tangent bundle of some compact surface



Proposition There exists $\alpha > 0$: For $f \in C^\infty(M)$, $\int f d\mu = 0$ then $\rho_f(t) = O(e^{-\alpha t})$

- We could consider $T: M \to M$ by $T \equiv \phi_{t=1}$
- Proof uses unitary representations of $SL(2,\mathbb{R})$

4. Horocycle flows

Let $\quad h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$

Let $\begin{cases} \psi_t : M \longrightarrow M \\ \psi_t(\{\pm I\}g\Gamma) = \{\pm I\}h_t g\Gamma \end{cases}$ "horocycle flow"

Proposition There exists $\alpha > 0$: For $f \in C^\infty(M)$, $\int f \, d\mu = 0$ then $\rho_f(t) = O(t^{-\alpha})$

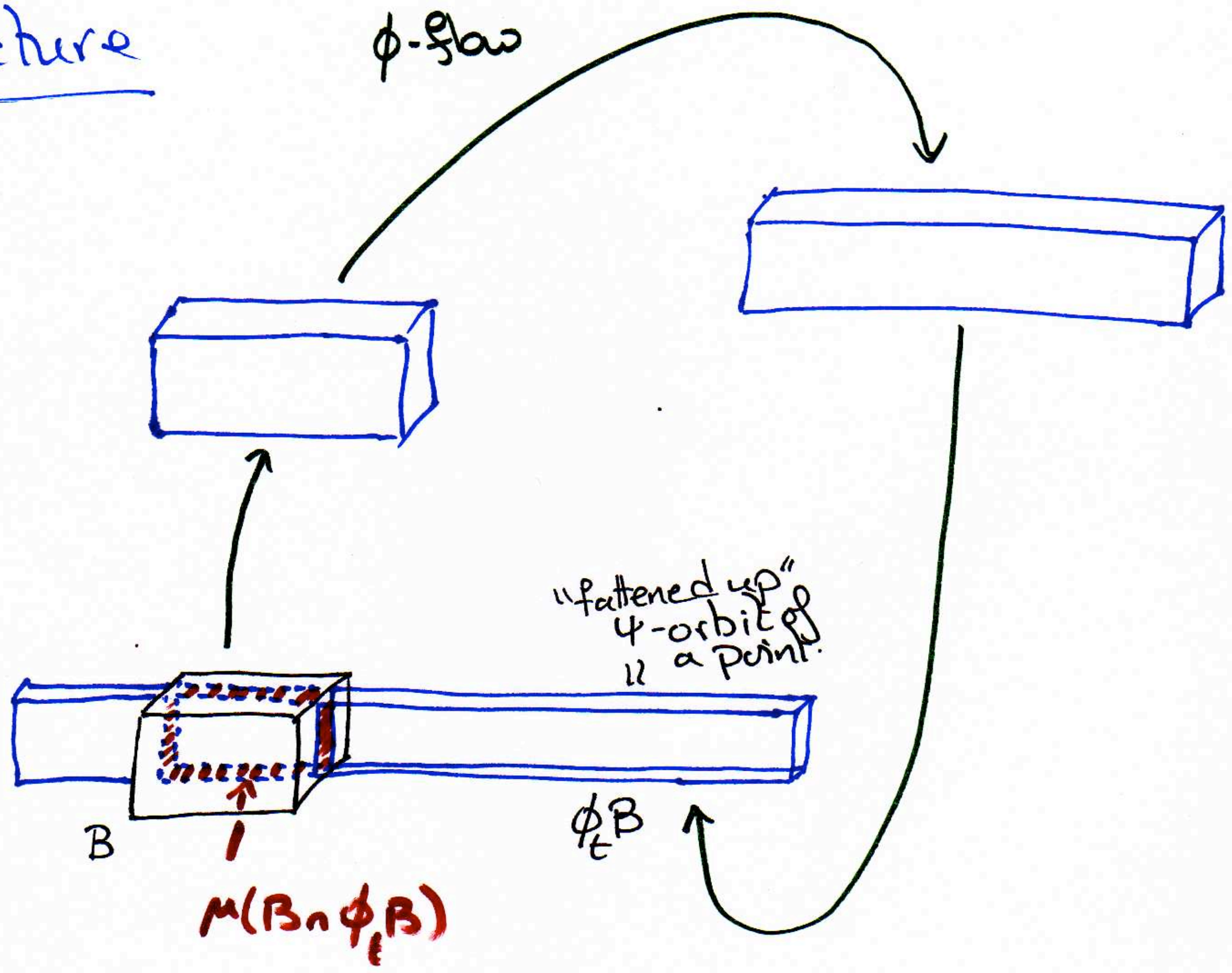
The key relationship between the flows is:

$$g_t h_s g_{-t} = h_{se^t} \quad \text{(matrix multiplication)}$$

$$\Rightarrow \quad \phi_t \psi_s \phi_{-t} = \psi_{se^t} \quad \text{(flows)}$$

(for $s, t \in \mathbb{R}$)

Picture



B = "small box"

Proposition There exists $\alpha > 0$: For $f \in C^\infty(M)$, $\int f d\mu = 0$ and <u>every</u> $x \in M$,

$$\frac{1}{T}\int_0^T f(\psi_s x)\,ds = O(s^{-\alpha})$$

This is sometimes called "Margulis trick"

5. Flaminio-Forni

<u>Question</u>. For which $f \in C^\infty(M)$ can we get faster convergence of Birkhoff averages?

<u>Theorem</u> (Flaminio-Forni)

There exist a countable family of linear maps

$$L_n : C^\infty(M) \longrightarrow \mathbb{R}, \quad n \geq 1,$$

such that for all $s \in \mathbb{R}$,

$$L_n(f \circ \varphi_s) = L_n(f), \quad f \in C^\infty(M), \ \int f \, d\mu = 0$$

and if $L_n(f) = 0$ for all $n \geq 1$ then:

(a) $$\frac{1}{T} \int_0^T f(\varphi_s x) \, ds = O(s^{-\alpha})$$

for all $x \in M$, $\alpha > 0$;

(b) f is a coboundary, ie,

$$\exists g \in C^\infty(M) : \quad f(x) = \frac{\partial}{\partial s} g(\varphi_s x)$$

(Proof uses unitary representations)

Open problems

(i) Relate the linear functionals $\{L_n\}$ to the speed of mixing $\rho_f(t)$ for the geodesic flow $\phi_t : M \to M$.

They should be related to the residues of the analytic extension of the Fourier transform $\hat{\rho}_f(z)$.

(ii) Generalize the Flaminio-Forni result to more general flows: eg Variable negative curvature geodesic flows? Anosov flows?