Numerical Computations in Smooth Ergodic Theory

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Dynamical systems

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Ingredients

- set X (interval, manifold, etc)
- map $T : X \to X$ (continuous, smooth, etc)

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Dynamical system

- iterates $\{T^n\}$ of map T
- models temporal process in time n

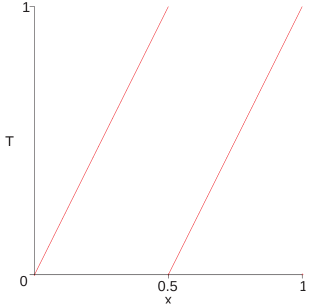
Observation

- temporal behaviour of system $\{T^n\}$ can be very complicated
- trajectories $\{T^n x\}$ might exhibit unpredictable, irregular behaviour

Example: doubling map

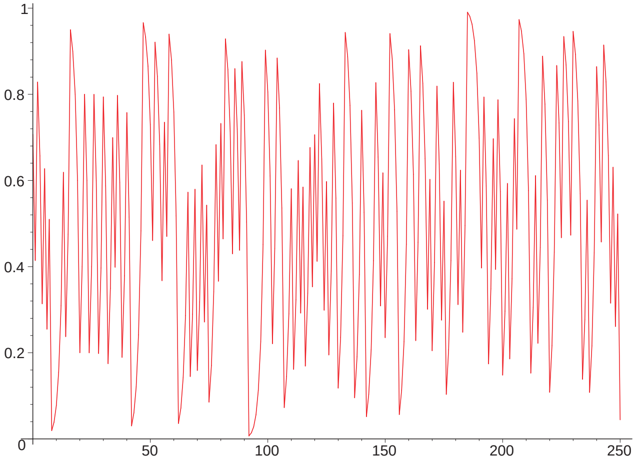
$$T : [0, 1] \to [0, 1]$$

$$T(x) = 2x \bmod 1$$



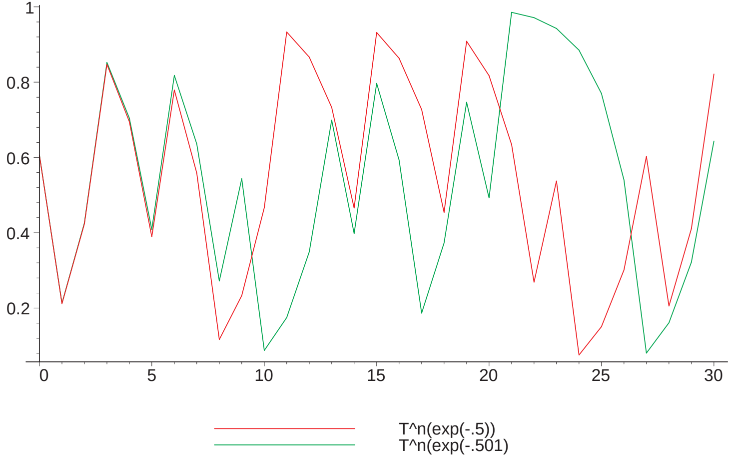
Example: doubling map...

Typical trajectory: 250 iterates of the point $1/\sqrt{2}$



Example: doubling map...

Sensitive dependence on initial conditions



Coping with chaos

Idea

Adopt probabilistic description

- assume: $\exists$ probability measure m on X
- for $A \subset X$ measurable consider

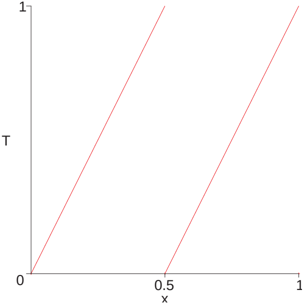
$$\mathrm{Prob}\{T^n x \in A\} = \mathrm{Prob}\{x \in T^{-n}A\} = m(T^{-n}A)$$

- more generally:
  consider stochastic process $\{f \circ T^n\}$ for given observable $f : X \to \mathbb{R}$

Example: doubling map again

$$T : [0,1] \to [0,1]$$

$$T(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}) \\ 2x - 1 & x \in [\frac{1}{2}, 1] \end{cases}$$



- notice that Lebesgue measure m on $[0,1]$ is invariant under T, i.e.

$$m(T^{-1}A) = m(A) \quad \forall A \text{ measurable}$$

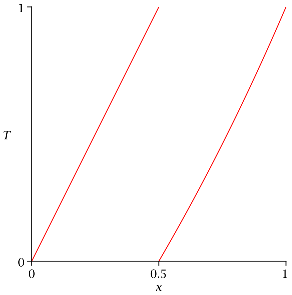
- thus:

$$\mathrm{Prob}\,\{T^n x \in A\} = \mathrm{Prob}\,\left\{x \in T^{-n}A\right\} = m(T^{-n}A) = m(A)$$

Another example

$$T : [0,1] \to [0,1]$$

$$T(x) = \begin{cases} 2x & x \in [0, 1/2) \\ \frac{6x-3}{4-x} & x \in [1/2, 1] \end{cases}$$



- notice that Lebesgue measure m on $[0,1]$ is <u>not</u> invariant under T
- thus:

$$\text{Prob}\,\{T^n x \in A\} = \text{Prob}\,\{x \in T^{-n}A\} = m(T^{-n}A) = ???$$

Smooth ergodic theory

- Given chaotic map $T : X \to X$ is there an *invariant measure*?
  I.e. a measure μ such that
  $$\mu(T^{-1}A) = \mu(A) \quad (\forall A \subset X \text{ measurable})$$
- If yes, is the measure μ well-behaved?
  I.e. *absolutely continuous w.r.t m*, i.e. of the form
  $$\mu(A) = \int_A \varrho \, dm$$
  for some m-integrable function $\varrho : X \to \mathbb{R}$.
- Is $T : (X, \mu) \to (X, \mu)$ ergodic? Mixing?

Basic tool: transfer operators

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With every smooth chaotic map T it is possible to associate a linear operator $\mathcal{L}$, known as *transfer operator*.

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Properties

- $\mathcal{L}$ is continuous linear operator on some infinite dimensional vector space (typically a functional Banach sapce)
- $\mathcal{L}$ is adjoint of *Koopman operator* $Ug = g \circ T$

$$\int_X \mathcal{L}f \cdot g \, dm = \int_X f \cdot Ug \, dm$$

- spectral properties of $\mathcal{L}$ yield insight into ergodic properties of T

Further properties of $\mathcal{L}$

- fixed point yields density of invariant measure:
  if $\mathcal{L}\varrho = \varrho$, $\varrho \geq 0$, then
  $$\mu(A) := \int_A \varrho \, dm$$
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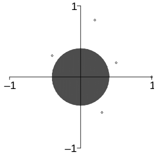

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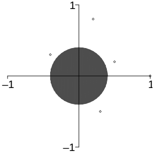

- spectral gap yields rate of decay of correlations:
  $$\int_X f \cdot g \circ T^n \, d\mu \sim |\lambda_2|^n \quad \text{as } n \to \infty$$
  where λ_2 is second largest eigenvalue of $\mathcal{L}$

How to calculate spectral data of $\mathcal{L}$?

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Idea

Use approximation scheme $\{P_n\}$ for $\mathcal{L} : B \to B$

$$P_n : B \to B \text{ projection}$$

$$\operatorname{rank} P_n = n$$

$$P_n f \to f \quad (\forall f \in B)$$

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Warning

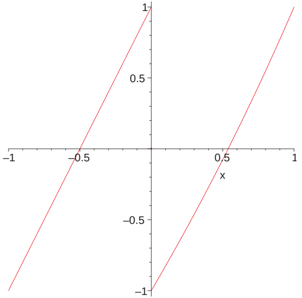
$\exists$ bounded operators L, approximation schemes $\{P_n\}$ such that

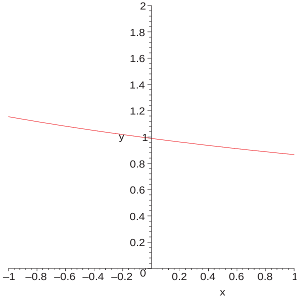
$$\sigma(L) = \text{unit disk}, \text{ but } \sigma(P_n \mathcal{L} P_n) = \{0\}$$

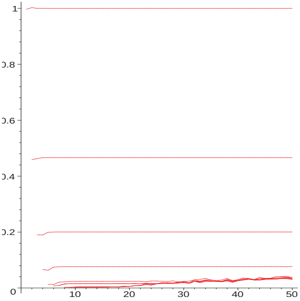
$$\sigma(L) = \{0, 1\}, \text{ but } \sigma(P_n \mathcal{L} P_n) = \{0, \frac{1}{2}, 1\}$$

Example: C^ω map

$$T(x) = \begin{cases} 2x+1 & x \in [-1,0) \\ \frac{13x-7}{7-x} & x \in [0,1] \end{cases}$$





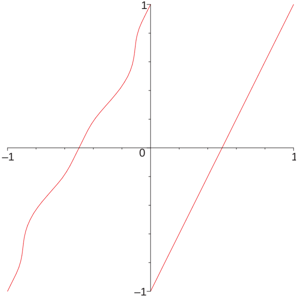


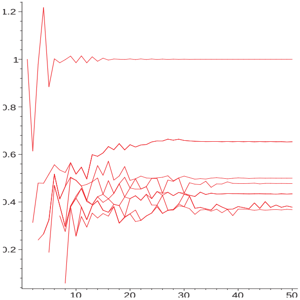
convergence of eigenvalues

rank 50 approximation to ϱ

Example: $C^\infty \setminus C^\omega$ map

$T =$ s.th. that doesn't fit on slide





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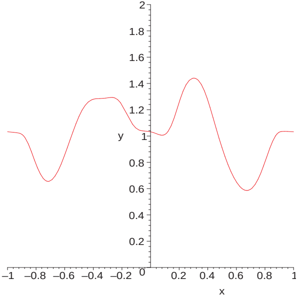

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Theorem (Newburgh 1951)

Let $L, L_n : B \to B$ be bounded and let $L_n \to L$ in operator norm. If $\sigma(L)$ is totally disconnected, then

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in the Hausdorff metric.

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If L is compact and $\{P_n\}$ is an approximation scheme, then

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Question

Is it possible to say something about the speed of convergence?