Recent Applications of Proof Theory in Metric Fixed Point and Ergodic Theory

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Corrected version Nov.20: a confused slide on the functional interpretation of weak compactness as well as a slide stating a bound on Browder's theorem have been deleted as the latter has been

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Goal: Additional information on the conclusion C:

- **Quantitative information: effective bounds.**

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Gödel's functional interpretation in five minutes

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- $A \leftrightarrow A^G$ by classical logic and **quantifier-free choice** in all types

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- x, y are tuples of **functionals of finite type** over the base types of the system at hand,

$$A^G \equiv \forall u \exists x\, A_G(u,x), \quad B^G \equiv \forall v \exists y\, B_G(v,y).$$

(G1) $P^G \equiv P \equiv P_G$ for atomic P

(G2) $(\neg A)^G \equiv \forall f \exists u\, \neg A_G(u, f(u))$

(G3) $(A \vee B)^G \equiv \forall u, v \exists x, y\, \big(A_G(u,x) \vee B_G(v,y)\big)$

(G4) $(\forall z\, A)^G \equiv \forall z, u \exists x\, A_G(z,u,x)$

(G5) $(A \rightarrow B)^G \equiv \forall f, v \exists u, y\, \big(A_G(u, f(u)) \rightarrow B_G(v,y)\big)$

(G6) $(\exists z A)^G \equiv \forall U \exists z, f\, A_G(z, U(z,f), f(U(z,f)))$

(G7) $(A \wedge B)^G \equiv \forall n, u, v \exists x, y\, (n{=}0 \rightarrow A_G(u,x)) \wedge (n{\neq}0 \rightarrow B_G(v,y))$
$\leftrightarrow \forall u, v \exists x, y\, \big(A_G(u,x) \wedge B_G(v,y)\big).$

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Monotone G (K. 1996) extracts Φ^* such that

$$\exists Y\ \left(\Phi^* \gtrsim Y \wedge \forall x\, A_G(x, Y(x))\right),$$

($\gtrsim$ some suitable notion 'bound', see later).
(see also: bounded interpretation, Ferreira/Oliva 2005).

Example: monotone convergence principle

Consider:

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall m \in \mathbb{N}\, (|r_n - r_{n+m}| < 2^{-k}).$$

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Solvable: Let $\tilde{g}(n) := n + g(n)$.

$$\exists n \le \max_{i \le 2^k}\{\tilde{g}^{(i)}(0)\}\ \forall i, j \in [n, n + g(n)]\, \left(|r_i - r_j| < 2^{-k}\right).$$

General logical metatheorems on the extractability of uniform bounds in analysis

- **Concrete Polish (P) and compact (K) metric spaces** are represented via $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$. Macros '$\forall x \in P, y \in K$'.

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- **Many abstract types of metric structures can be added as atoms:** metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, ... spaces or $\mathbb{R}$-trees X : add **new base type X**, all **finite types over $\mathbb{N}, X$** and a new **constant d_X** representing d etc.

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  Crucial for uniformity of bounds: **no separability assumptions!**

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic, normed, Hilbert . . . space.

(Some restriction on extensionality necessary)

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$x^{\mathbb{N}} \gtrsim^a_{\mathbb{N}} y^{\mathbb{N}} :\equiv x \geq y$$
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Example:

$$f^* \gtrsim^a_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d(a,x) \to f^*(n) \geq d(a, f(x))].$$

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$f : X \to X$ is **nonexpansive (n.e.)** if $d(f(x), f(y)) \leq d(x, y)$.

Then $\lambda n.n + b \gtrsim^a_{X \to X} f$, if $d(a, f(a)) \leq b$.

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K. Trans. AMS 2005, Gerhardy/K. Trans AMS 2008.

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, d, \ldots]$ proves

$$\forall x \in P \forall y \in K \forall z \in X \forall f : X \to X \big(f \text{ n.e.} \to \exists v \in \mathbb{N}\, A_\exists\big),$$

then one can extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $x \in P, b \in \mathbb{N}$

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For **normed** and **Hilbert** spaces with $b \geq \|z\|$ (K. TAMS 2005).

Proof Mining in Ergodic Theory

Let X be a **Hilbert space**, $f : X \to X$ **linear and nonexpansive**.

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on logical metatheorem discussed above:

Theorem (K./Leuştean, to appear in Ergodic Theor. Dynam. Syst.)

Assume that X is a uniformly convex Banach space, η is a modulus of uniform convexity and $f : X \to X$ is a nonexpansive linear operator. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \le n} h(i).$$

Corollary (K./Leuştean, to appear in Ergodic Theor. Dynam. Syst.)

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Discussion: The Hilbert space case has been treated (again based on our metatheorem) prior by Avigad-Gerhardy-Towsner (TAMS to appear). However, the bound obtained by Avigad et al. is less good and matches our bound only in the special case of isometric f.

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‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Projections and Weak Compactness without separability

(K.2009, to appear in: Festschrift for G. Mints)

$\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle]$ does not have nontrivial comprehension over X-type objects but proves (using countable choice for X-objects) **schematically**

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- for linear functionals $L : X \to \mathbb{R}$ with **definable graph** the Riesz representation theorem,
- the **weak compactness of $B_1(0)$** (here only countable choice for arithmetical formulas needed and restricted induction).

A theorem of F.E. Browder

Using projection to the set of all fixed points of a nonexpansive mapping $U : X \to X$ with $U|_{B_1(0)} : B_1(0) \to B_1(0)$ (X Hilbert space) and weak compactness, Browder showed in 1967:

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Theorem[F.E. Browder 1967]: For $n \in \mathbb{N}, v_0 \in B_1(0)$ let u_n be the unique fixed point of the contraction $U_n(x) := (1 - \frac{1}{n})U(x) - \frac{1}{n}v_0$. Then (u_n) converges towards the fixed point of U that is closest to v_0.

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Corollary by Metatheorem: There is a functional $\Phi(k, g)$ (definable by primitive recursion and bar recursion of lowest type) such that

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k, g)\, \forall i, j \in [n; n + g(n)]\, (\|u_i - u_j\| < 2^{-k}).$$

Note that Φ does not depend on U, v_0 or X!

Nonlinear Generalizations of the Mean Ergodic Theorem

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Theorem (Wittmann 1992)

$(X, \langle \cdot, \cdot \rangle)$ Hilbert space, $C \subseteq X$ closed convex, $f : C \to C$ nonexpansive. (λ_n) in $[0, 1]$ with

$$\lim \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n \text{ divergent}, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \text{ convergent.}$$

Consider the Halpern iteration starting from $x \in X$

$$x_{n+1} := \lambda_{n+1} x + (1 - \lambda_{n+1}) f(x_n).$$

If f has fixed points, then (x_n) converges to the unique fixed point closest to x.

Remark

If f is linear and $\lambda_n := \frac{1}{n+1}$, then

$$x_n = \frac{1}{n+1} \sum_{i=0}^{n} f^i x.$$

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Theorem (Leustean 2007)

Let X be just normed and T, C as above and $\lim \lambda_n = 0$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ convergent and $\sum_{n=1}^{\infty} \lambda_n$ divergent with moduli α, β, θ resp. Then

$$\forall \varepsilon(0,2)\, \forall n \geq \Phi(\alpha, \beta, \theta, M, \varepsilon)\ (\|x_n - f(x_n)\| < \varepsilon),\ \textit{where}$$
$$\Phi(\alpha, \beta, \theta, M, \varepsilon) := \max\left\{\theta\left(\beta\left(\tfrac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\tfrac{8M}{\varepsilon}\right)\right\rceil\right), \alpha\left(\tfrac{\varepsilon}{4M}\right)\right\},$$

with $M \geq \|x_n\| + \|x\| + \|Tx\|$ for all $n \in \mathbb{N}$.

- Next step: extraction of **effective rate of metastability** for (x_n). Proof similar (but more involved) than that of Browder's theorem

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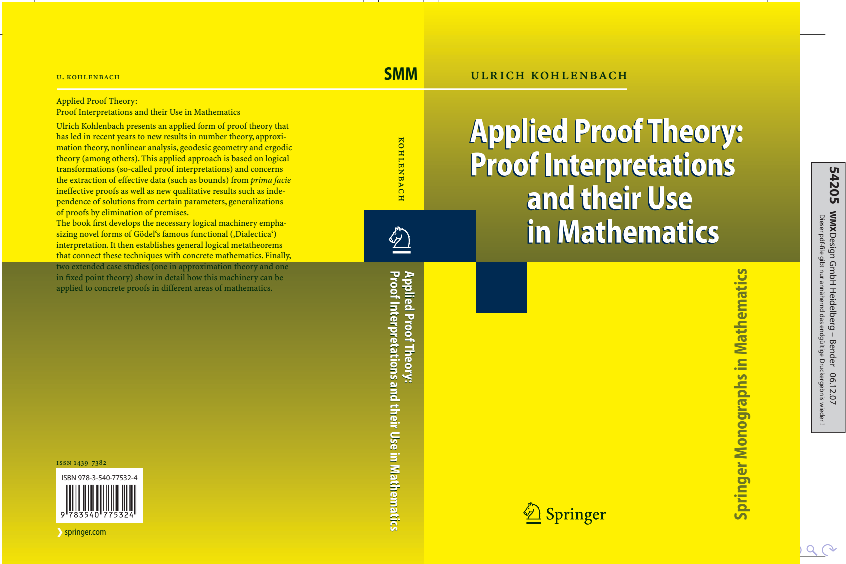
- Future research: quantitative version of Baillon's nonlinear ergodic theorem.

Further applications of proof theory to analysis

- Numerous further applications in metric fixed point and ergodic theory (Avigad, Briseid, Gerhardy, Lambov, Leustean, Towsner, K.) published in: J.Math.Anal.Appl.(4), Trans. Amer. Math. Soc. (3), Nonlinear Analysis (2), Numer.Funct.Anal.Opt.(2), Fixed Point Theory (2), Intern. J. of Math. and Statistics (2), Ergod. Theor. and Dynam. Syst. (2), Proc. Fixed Point Theory (1), J. Nonlinear and Convex Analysis (1)
- Applications to best Chebycheff and L_1-approximation: improved resp. first at all bounds on strong unicity (K., K./Oliva: Numer.Funct.Anal.Opt., Steklov proc.,...)


U. KOHLENBACH
Applied Proof Theory:
Proof Interpretations and their Use in Mathematics
Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from prima facie ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.
The book first develops the necessary logical machinery emphasizing novel forms of Gödel‘s famous functional (‚Dialectica‘) interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.
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