Studying equivalences of transition systems with algebraic tools

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Abstract

The aim of this paper is twofold. On one side we will characterize observational equivalences (simulation, bisimulation) in an algebraic framework. On the other side we will deduce by this algebraic framework a new equivalence (the skeleton equivalence), which is an equivalence situated between trace equivalence and equality of languages.

In order to characterize simulation equivalence we will define a monad on the category of transition systems.

We introduce a category of algebras to characterize bisimulation. This category turns out to be the "stone dual" of the category of transition systems. Moreover this category of algebras seems to be a natural framework to reason about bisimulation equivalence; bisimulation corresponds to subalgebras isomorphims and the minimal transition system in a bisimulation class corresponds to the minimal subalgebra of a given algebra.

Eventually the notion of minimal subalgebra will, roughly speaking, be factorized as the boolean completion of the "skeleton" of an algebra, so that the concept of skeleton equivalence naturally arises.

Contents

1 Introduction

The problem of equivalence is one of the most important problems in the theory of transition systems.

Several equivalences have been proposed following the possible ideas of what a transition system is intended to modelize.

However even when these equivalences are easy to define, they are in general difficult to use because of their syntactical nature.

For this reason abstract tools have been introduced in order to characterize these equivalences as mathematically as possible.

For bisimulation equivalence, one of the most successful approaches is the Hennessy-Milner logic [HM]. A more algebraic approach has been developped by Arnold and Dicky, [Ar] by using algebras that can be intuitively seen as the Lindembaum algebras for the Hennessy-Milner logic.

Our approach for bisimulation will be strictly related to the Arnold's approach.

1.1 An algebraic framework for bisimulation

Let's first clarify what we mean by an "algebraic framework" for bisimulation.

An algebraic framework should associate a category of algebras to the category of transition systems by means of a bijective functor (so that any transition system has a corresponding unique algebra and any algebra has a corresponding unique transition system).

Moreover one would like this association to transform properties of transition systems into algebraic properties, so that problems in the category of transition systems can be treated by pure algebraic means.

Now maybe the most natural notion in algebra is that of subalgebra. Hence subalgebra would be the ideal candidate to represent the concept of bisimulation: we are thinking of a theorem like the following:

- "Two transition systems T and T' are equivalent by bisimulation if and only if their associated algebras have an isomorphic subalgebra."

This is indeed the result we will prove.

The category of algebras we will introduce to prove this result is related to the category of transition systems by a Stone duality.

Stone duality makes it possible to establish an equivalence between categories having a very different structure, for example between categories of algebras and categories of topological spaces [John], or between categories of domains and categories of algebras and logics [Ab, EM]. Moreover it is our belief that the algebras given by a Stone duality are in a certain way "canonical"; in our case canonical means that the algebra associated to a transition system is as close as possible to the structure of the system (roughly speaking it is the space of ultrafilters on the systems).

A further benefit of this algebraic framework is that it allows the definition of a new equivalence: the skeleton equivalence. The skeleton of an algebra is obtained by considering the closure of a subset of the carrier under just one algebraic operation. The equivalence is then defined by stipulating that two systems are equivalents if the skeletons of their respective algebras are isomorphic. Skeleton equivalence is rather weak, because it lies between the trace equivalence and the equality of languages.

1.2 The categorical notion of algebra

Category theory gives a very abstract definition of algebra, by means of the notion of "being monadic over ..." [McL], [AM]. For example any variety (in the sense of universal algebra) can be seen as a category which is monadic over **Set**.

In particular the category of complete sup semilattice can be identified with the Eilenberg-Moore category given by the "powerset" monad $M = (\wp, \{-\}, \bigcup)$.

Indeed one can generalize powerset-like monads on categories which are "enriched" over the category **Set**; in this case one should think in terms of algebras whose carrier has an additional structure.

In our context, since the category of transition systems can be thought as an enriched category over **Set**, we will use these categorical means in order to define a "sup semilattice completion" of a transition system; the idea here is that the completion of a system T represents the space of all possible simulations on T.

We will then characterize simulation as follows:

- "The transition system T is simulated by T' if and only if there exists a transition system morphism from T to the sup semilattice completion of T'".

Interestingly it turned out that this functor characterizing simulation was already known in the community because it had been studied by N.Klarlund and F.B.Schneider [KS],[ADN] in order to study the problem of the canonical nondeterministic automaton associated to a rational language.

The fact that two rather different problems can be solved by using the same (abstract) tool seems to be quite intriguing and deserving further research.

2 Transition systems and action algebras

2.1 some remarks about boolean algebras

A *Complete Atomic Boolean algebra* (shortly **CBA**) is a Boolean algebra A which is *complete*, i.e. each subset has an inf and a sup and is *atomic*, i.e. there exists a (non empty) subset $\mathrm{At}(A)$ of A such that the following properties hold:

AT1 For any elements $v \in A, a \in \mathrm{At}(A)$ if $a \not\leq v$ then $a \wedge v = 0$.

AT2 For any elements $v \neq 0$ of A there exists a $a \in \mathrm{At}(A)$ such that $a \leq v$.

A morphism between two CBA's A and A' is a map from A to A' which preserves the structure, i.e. commutes to sups, infs, complementation.

There is a well known representation property for CBA's [Hal] which is

Proposition 1 *Each* **CBA** *A is isomorphic to the boolean algebras of the power set of* $\mathrm{At}(A)$.

Moreover each boolean algebra which is a power set of a set is complete and atomic; hence we can think of a **CBA** morphism ϕ between A and A' as a **CBA** morphism from $\wp(\mathrm{At}(A))$ to $\wp(\mathrm{At}(A'))$ which turns out to be the inverse image f^{-1} of a set theoretic map $f : \mathrm{At}(A') \to \mathrm{At}(A)$.

It follows that we can define $\phi_* : \mathrm{At}(A') \to \mathrm{At}(A)$ as the set theoretic *underlying map* of ϕ.

On the other hand, given ϕ_* we can construct $\phi_*^+ : \wp\mathrm{At}(A') \to \wp\mathrm{At}(A)$ which is no more than the extension of ϕ to the subset of $\mathrm{At}(A')$ (that is $\phi_*^+(v) = \{\phi_*(a) | a \in v\}$). We have then

Proposition 2 *The maps ϕ_*^+, ϕ form a Galois pair i.e. $\phi_*^+(u) \subseteq v \iff u \subseteq \phi(v)$.*

A typical and useful consequence of proposition 2 is that $v \subseteq \phi(\phi_*^+(v))$ and $\phi_*^+(\phi(u)) \subseteq u$.

2.2 Categories of algebras and transition systems

Let A be a **CBA** and X a set; a *linear action* of X on A is then given by a map $\alpha : X \times A \to A$ (we write $x.v$ instead of $\alpha(x, v)$) such that:

- $x.0 = 0$
- $x.\bigvee V = \bigvee_{v \leq V}(x.v)$

The *category of actions of X over complete atomic boolean algebras* (category denoted as $\mathcal{AL}$) has as objects pairs (A, α), where A is a **CBA** and α is an action of X over A and as arrows $\phi : (A, \alpha) \to (A', \alpha')$ where ϕ is a **CBA** morphism between A and A' which satisfies the inequality

$$x.\phi(v) \leq \phi(x.v)$$

The *category of transition systems over an alphabet X* (which we denote as $\mathcal{TS}$) has as objects pairs $T = (S, T)$ (we use the same letter T to indicate the set of transitions and the transition system) where S is the set of *states* and $T \subseteq S \times X \times S$ is the set of *transitions* whose elements we denote as $s \xrightarrow{x} s'$. An arrow f from (S, T) to (S', T') is a set theoretic map $f : S \to S'$ such that

$$s \xrightarrow{x} s' \in T \Rightarrow f(s) \xrightarrow{x} f(s') \in T'$$

We can define a functor **Ac** from $\mathcal{TS}$ to $\mathcal{AL}$ in the following way:
$\mathbf{Ac}(T) = (\wp(S), \alpha)$ where $\alpha(x, v) = \{s_1 \in S | \exists s_2 \in v \text{ such that } s_1 \xrightarrow{x} s_2\}$
$\mathbf{Ac}(f) = f^{-1}$

Lemma 1 **Ac** *is a (contravariant) functor*

On the other side we can define a functor $\mathbf{Ts} : \mathcal{AL} \to \mathcal{TS}$ as follows:
$\mathbf{Ts}(A, \alpha) = (\mathrm{At}(A), T_A)$ where $a_1 \xrightarrow{x} a_2 \in T_A$ iff $a_1 \leq x.a_2$
$\mathbf{Ts}(\phi) = \phi_*$ (the underlying map defined in section 1)

Lemma 2 **Ts** *is a (contravariant) functor*

Proposition 3 $\mathbf{Ts}, \mathbf{Ac}$ *define a (contravariant) equivalence between the categories $\mathcal{TS}, \mathcal{AL}$*

Notational Convention: Unless otherwise stated we will use the following notations:

$a, a', b \ldots$ for atoms of an algebra; $v, v', \ldots$ for arbitrary elements of an algebra. Sometimes we will use set theoretical symbols instead of order theoretical ones ($a \in v$ instead of $a \leq v$, etc.); this usage is justified by proposition 1.

$T, T', T_1 \ldots$ will denote transition systems $(S, T), (S', T'), (S_1, T_1) \ldots$. Similarly $A, A', \ldots$ will abbreviate $(A, \alpha), (A', \alpha'), \ldots$.

Categorical notations are as in [McL].

2.3 Categorical properties of $\mathcal{TS}$

We list here some categorical properties of $\mathcal{TS}$ which will be useful in the rest of the paper:

- The category $\mathcal{TS}$ has small limits; This can be proved by showing that $\mathcal{TS}$ has small products and equalizers:
- the *product* of a family $(T_i)_{i \in I}$ is the transition system $\times T = (\times S, \times T)$ where $\times S$ is the set theoretical product of the family $(S_i)_{i \in I}$ i.e.

  $$\{V : I \rightarrow \bigcup_{i \in I} T_i | \forall i \in I (V(i) \in T_i)\}$$

  $V_1 \xrightarrow{x} V_2 \in \times T$ iff for any $i \in I$ $V_1(i) \xrightarrow{x} V_2(i) \in T_i$
- The *equalizer* of $f, g \in \mathcal{TS}(T, T')$ is given by the system T_0 such that $S_0 \subseteq S$ and $s \xrightarrow{x} s' \in T_0$ iff $f(s) \xrightarrow{x} f(s') \in T'$ and $g(s) \xrightarrow{x} g(s') \in T'$. The map which "equalizes" f and g is of course the inclusion map from S_0 to S.
- The category $\mathcal{TS}$ has *sums* as well. The sum of the family $(T_i)_{i \in I}$ is the transition system $+T = (+S, +T)$ where

  $+S$ is the disjoint union of the family $(S_i)_{i \in I}$

  $(s, i) \xrightarrow{x} (s', j) \in +T$ iff $i = j$ and $s \xrightarrow{x} s' \in T_i$
- The *terminal object* in $\mathcal{TS}$ is the system $(\{*\}, \{* \xrightarrow{x} * | x \in X\})$
- The *initial object* in $\mathcal{TS}$ is the system $(\{*\}, \emptyset)$
- Given two transistion systems T, T' we can construct the *weak exponential* $T^{T'}$ as follows:

  The set of states of $T^{T'}$ is the set $\mathcal{TS}(T', T)$.

  Given $f, g \in \mathcal{TS}(T', T)$ the transition $f \xrightarrow{x} g$ is a transition in $T^{T'}$ iff for all $s \in S$ $f(s) \xrightarrow{x} g(s) \in T'$

Remark : The object $T^{T'}$ is not an exponential in categorical terms; for if it were, the evaluation map $\epsilon(T', T) : T^{T'} \times T' \rightarrow T$ associating $(f, s) \xrightarrow{x} (g, s') \in T^{T'} \times T'$ to its value $f(s) \xrightarrow{x} g(s')$ should esists. However this transition is in general not in T since the definition of product and weak exponential only allows for $f(s) \xrightarrow{x} g(s)$ and $g(s) \xrightarrow{x} g(s')$ in T. Hence in order to have a true exponential it is sufficient to require a transitivity property of transition, i.e. if $s_1 \xrightarrow{x} s_2$ and $s_2 \xrightarrow{x} s_3$ are in T then $s_1 \xrightarrow{x} s_3 \in T$.

3 Stone duality between $\mathcal{TS}$ and $\mathcal{AL}$

In this section we show that the previously proved duality between $\mathcal{TS}, \mathcal{AL}$ is a "Stone like duality".

By this we mean that the functor **Ts** (resp **Ac**) can be defined in terms of the hom functor $\mathcal{TS}(-, \Omega_{\mathcal{TS}})$ (resp $\mathcal{AL}(-, \Omega_{\mathcal{AL}})$) where $\Omega_{\mathcal{TS}}$ (resp $\Omega_{\mathcal{AL}}$) is a particular element of $\mathcal{TS}$ (resp $\mathcal{AL}$).

To be precise we should think of the functor $\mathcal{TS}(-, \Omega_{\mathcal{TS}}) : \mathcal{TS}^{op} \to \mathbf{Set}$ (resp $\mathcal{AL}(-, \Omega_{\mathcal{AL}}) : \mathcal{AL}^{op} \to \mathbf{Set}$) as an enriched functor $\mathcal{TS}(-, \Omega_{\mathcal{TS}}) : \mathcal{TS}^{op} \to \mathcal{AL}$ (resp $\mathcal{AL}(-, \Omega_{\mathcal{AL}}) : \mathcal{AL}^{op} \to \mathcal{TS}$).

This enriched functor $\mathcal{TS}(-, \Omega_{\mathcal{TS}})$ (resp $\mathcal{AL}(-, \Omega_{\mathcal{AL}})$) which we shall define later is the functor we refer to in this paper.

The simplest, and for our aim useful, example of Stone duality may be the following: lets consider the (contravariant) functor $\wp : \mathbf{Set} \to \mathbf{CBA}$ which maps a set to its power set and a set theoretic map to its inverse image. By using the isomorphism between a subset v of S and its characteristic map $\chi_v : S \to \{0, 1\}$ it follows that $\wp(S) \simeq \mathbf{Set}(S, \{0, 1\})$. Moreover for $f : S' \to S$ we have

$$\begin{aligned} \chi_{\wp(f)(v)} &= \chi_{f^{-1}(v)} \\ &= \chi_v \circ f \\ &= \mathbf{Set}(f, \{0, 1\})(v) \end{aligned}$$

which allows us to "eliminate" the functor $\wp$ in favour of the functor $\mathbf{Set}(-, \{0, 1\})$; formally this means that the map $\Psi(S)$ which associates $v \subseteq S$ to $\chi_v : S \to \{0, 1\}$ is indeed a natural isomorphism $\Psi : \wp \to \mathbf{Set}(-, \{0, 1\})$.

The previous example is helpful in order to find the objet we are looking for; indeed let $\Omega_{\mathcal{TS}}$ be the following transition system

$$\begin{tikzcd} 0 \arrow[loop left, "X"] \arrow[r, bend left, "X"] & 1 \arrow[l, "X"] \arrow[loop right, "X"] \end{tikzcd}$$

That is $\Omega_{\mathcal{TS}} = (\{0, 1\}, \{s_1 \stackrel{x}{\to} s_2 | s_1, s_2 \in \{0, 1\}, x \in X\})$ We can now prove the aforesaid property:

Proposition 4 *The functors* **Ac** *and* $\mathcal{TS}(-, \Omega_{\mathcal{TS}})$ *are naturally isomorphic.*

Proof: This means that the functor $\mathcal{TS}(-, \Omega_{\mathcal{TS}})$ associates a transition system T to an algebraic structure. This can be proved as follows:

1 Any set theoretic map $f : S \to \{0, 1\}$ is in $\mathcal{TS}(T, \Omega_{\mathcal{TS}})$, hence $f : \mathcal{TS}(T, \Omega_{\mathcal{TS}})$ is the characteristic map χ_V of a subset V of S

2 The set of characteristic maps on S has an algebraic structure: for a boolean operation $\omega \in \{\bigcup, \bigcap\}$ we put

$$\omega(\chi_{V\,i})_{i \in I}(s) = 1 \iff s \in \omega(\chi_i^{-1}(1))_{i \in I}.$$

Similarly the complementation is defined by $\neg\chi_v(s) = 1$ if and only if $\chi_v(s) = 0$

We only have to define the actions, i.e. $x.\chi_V$; we put

$$x.\chi_V(s) = 1 \ \text{ iff } \ s \in x.(\chi_V^{-1}(1))$$

It is easy to verify that $x.\chi_V = \chi_{x.V}$.

The isomorphism is then $\Psi : \mathcal{TS}(-, \Omega_{\mathcal{TS}}) \rightarrow \mathbf{Ac}$ defined by $\Psi(T)(\chi_V) = V = [\chi_V^{-1}](1)$; it is natural since for any $f : T' \rightarrow T, V \subseteq S$ the following identities hold:

$$\begin{array}{rcll} [\Psi(T') \circ \mathcal{TS}(f, \Omega_{\mathcal{TS}})](\chi_V) & = & [\mathbf{Ac}(f) \circ \Psi(T)](\chi_V) & \text{iff} \\ \Psi(T')(\chi_V \circ f) & = & f^{-1}(V) & \text{iff} \\ f^{-1}(\chi_V^{-1}(1)) & = & f^{-1}(V) & . \end{array}$$

Which completes the proof.

■

In order to find the object $\Omega_{\mathcal{AL}}$ we consider the following algebra:

$$\begin{tikzcd} 1 \arrow[d, "x"'] \\ 0 \arrow[loop right, "x"] \end{tikzcd}$$

That is $\Omega = (\{0, 1\}, \alpha)$ where α is defined by $x.a = 0$ for $a \in \{0, 1\}$

Proposition 5 *The functors* $\mathbf{Ts}$ *and* $\mathcal{AL}(-, \Omega_{\mathcal{AL}})$ *are naturally isomorphic.*

Proof: We first remark that

$$\mathrm{HOM}_{\mathcal{AL}}((A, \alpha), \Omega_{\mathcal{AL}}) = \mathrm{HOM}_{\mathbf{CBA}}\ (A, \{0, 1\})$$

since each morhism ϕ in $\mathbf{CBA}$ $(A, \{0, 1\})$ satisfies the inequality

$$x.\phi(v) \leq \phi(x.v)$$

Note then that the underlying map of each morphism of complete boolean algebra ϕ whose range is the algebra $\{0, 1\}$ has the shape

$$\phi_* \ : \{1\} \rightarrow \mathrm{At}(A)$$

so that there exists an atom $a = \phi_*(1)$ such that $\phi(u) = 1$ if and only if $u \geq a$.

Hence we can write such a morphism ϕ as $\uparrow a$ where $a = \phi_*(1)$, by putting

$$(\uparrow a)(u) = 1 \text{ if and only if } a \leq u$$

then we deduce an isomorphism between $\mathrm{HOM}_{\mathbf{CBA}}\ (A, \{0, 1\})$ extensionally ordered and $\{\uparrow \alpha \mid \alpha \in \mathrm{At}(A)\}$ ordered by inclusion.

Hence we define the transition system $(\mathcal{AL}((A,\alpha),\Omega_{\mathcal{AL}}),T_{(A,\alpha)})$ by putting

$$\uparrow a \xrightarrow{x} \uparrow a' \in T_{(A,\alpha)} \Longleftrightarrow a \leq x.a'$$

We can then prove that the map $\iota : \mathbf{Ts} \to \mathcal{AL}(-,\Omega_{\mathcal{AL}})$ defined by $\iota(A)(a) = \uparrow a$ is a natural isomorphism.

In the first place it is a transition system isomorphism since

$$a \xrightarrow{x} a' \in T_A \Longleftrightarrow \uparrow a \xrightarrow{x} \uparrow a' \in T_{(A,\alpha)}$$

Moreover it is natural since for any $\phi : A' \to A, a \in \mathrm{At}(A)$ we have

$$\begin{array}{rrcll} & [\iota(A') \circ \mathbf{Ac}(\phi)](a) & = & [\mathcal{AL}(\phi,\Omega_{\mathcal{AL}}) \circ \iota(A)](a) & \text{iff} \\ & \uparrow(\phi_*(a)) & = & (\uparrow a) \circ \phi & \text{iff} \\ \forall v \in A' & (\phi_*(a) \leq v & \Longleftrightarrow & a \leq \phi(v)) & . \end{array}$$

the last line being true because of proposition 2 ∎

We recall that a *generator* in a category $\mathcal{C}$ is given by an object C of $\mathcal{C}$ such that for any C_1, C_2 in $\mathcal{C}$ and $f, g : C_1 \to C_2$ there exists $p : C \to C_1$ such that $f \neq g$ implies $f \circ p \neq g \circ p$. The notion of *cogenerator* is dually defined. As a corollary of the Stone duality just proven we have:

Proposition 6 *Both categories $\mathcal{TS}, \mathcal{AL}$ have generator and cogenerator*

Proof: The generator of $\mathcal{TS}$ is $\mathbf{Ts}(\Omega_{\mathcal{AL}})$, the cogenerator of $\mathcal{TS}$ is $\Omega_{\mathcal{TS}}$. Dually the generator and cogenerator of $\mathcal{AL}$ are respectively $\mathbf{Ac}(\Omega_{\mathcal{TS}})$ and $\Omega_{\mathcal{AL}}$ ∎

4 Simulation as a monad on $\mathcal{TS}$

Given two transition systems $T = (S,T)$, $T' = (S',T')$ a *simulation* between T and T' is a relation $\mathcal{R} \subseteq S \times S'$ such that

$\mathcal{R}1$ For any $s \in S$ there exists $s' \in S'$ such that $(s,s') \in \mathcal{R}$

$\mathcal{R}2$ For any $s_1 \xrightarrow{x} s_2 \in T$ if $(s_1,s_1') \in \mathcal{R}$ then there exists $s_2' \in S'$ such that $s'_1 \xrightarrow{x} s'_2 \in T'$ and $(s_2,s_2') \in \mathcal{R}$

T and T' are *equivalent by simulation* if there exists a simulation R between T and T' and a simulation R' between T' and T. In case $R' = R^{-1}$ T and T' are said to be *equivalent by bisimulation.*

The notions of simulation and bisimulation arise from the study of the "observationally equivalence" of systems; in this sense the alphabet represents the set of all possible actions that a system can perform. The idea is then that an external observer can discriminate between the two systems only by proposing an experiment to which the two systems will respond differently, i.e. will produce different actions. If T and T' are in bisimulation such an experiment can never be proposed.

Indeed any notion of "observational equivalence" must contain the equivalence induced by bisimulation, hence bisimulation is the finest observational equivalence. The equivalence induced by simulation (i.e. T, T' are equivalent iff there exists a simulation between T and T' and a simulation between T' and T) is weaker; the following is an example of two systems which are equivalent in this sense but are not in bisimulation:

$$\begin{tikzcd}
T_1 & 0 \arrow[loop above, "x"] \arrow[rr, "x"] & & 1 \\
T_2 & 0 \arrow[loop above, "x"] & & 1
\end{tikzcd}$$

We will study simulation in terms of a monad[1] in the category $\mathcal{TS}$.
Let the functor $\mathbf{Sm} : \mathcal{TS} \to \mathcal{TS}$ be defined as follows:

1 $\mathbf{Sm}(S, T) = (\wp^+(S), T^+)$ where $\wp^+(S)$ is the set of non empty subsets of states of S and $V_1 \xrightarrow{x} V_2 \in T^+$ iff for any $s_1 \in V_1$ there exists $s_2 \in V_2$ such that $s_1 \xrightarrow{x} s_2 \in T$.

2 $\mathbf{Sm}(f) = f^+$ is the extension of $f : (S, T) \to (S', T')$ to the subset of S (cfr section 1)

We immediately see that $\mathbf{Sm}$ is a functor; moreover we can define a monad $(\mathbf{Sm}, \eta, \mu)$ by putting
$\eta(S, T)(s) = \{s\}$ and
$\mu(S, T)(\mathcal{V}) = \bigcup_{V \in \mathcal{V}} V$

Lemma 3 $(\mathbf{Sm}, \eta, \mu)$ *is a monad on* $\mathcal{TS}$.

Let the Kleisli category[2] of the monad $\mathbf{Sm}$ on $\mathcal{TS}$ be denoted by $K_{\mathbf{Sm}}$

Proposition 7 *The following are equivalent:*

- *There exists a simulation between* T *and* T'
- *There exists an arrow between* T *and* T' *in* $K_{\mathbf{Sm}}$

Proof: Given a simulation $\mathcal{R} \subseteq S \times S'$ we define $\rho : S \to \wp^+(S')$ by $\rho(s) \stackrel{\text{def}}{=} \{s' | (s, s') \in \mathcal{R}\}$. It is easily seen that ρ is a transition system map.

On the other hand an arrow $\rho : T \to \mathbf{Sm}(T')$ seen as $\{(s, s') | s' \in \rho(s)\}$ is a relation between S, S' that satisfies $\mathcal{R}1$ since the empty set is not in $\wp^+(S)$ and satisfies $\mathcal{R}2$ because it is a transition system map ∎

[1] A monad on a category $\mathcal{C}$ is given by a functor $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ and two natural transformation $\eta : 1 \to \mathbf{F}$ and $\mu : \mathbf{F}^2 \to \mathbf{F}$ such that $\mu \circ \eta\mathbf{F} = 1_{\mathbf{F}} = \mu \circ \mathbf{F}\eta$ and $\mu \circ \mu\mathbf{F} = \mu \circ \mathbf{F}\mu$

[2] Given a monad $(\mathbf{F}, \eta, \mu)$ in a category $\mathcal{C}$, the Kleisli category of $\mathbf{F}$ on $\mathcal{C}$ has as objects the same objects as $\mathcal{C}$ and as arrows from C to C' all the arrows in $\mathcal{C}$ from C to $\mathbf{F}C$. The composition $f * g$ of two arrows $g : C_1 \to C_2$, $f : C_2 \to C_3$ in the Kleisli is defined by $\mu(C_3) \circ \mathbf{F}(f) \circ g$; the identity arrow $1_K(C)$ from C to C in the Kleisli is defined as $\eta(C)$.

An important principle in the theory of transition systems is the so called "compositionality" principle. A theory is compositional if it allows to prove properties of a system by proving properties of its subsystems. We can then look for categorical constructions which preserve the simulation relation.

Lemma 4 *$K_{\mathbf{Sm}}$ has sums and weak products*[3]

Proof: Existence of sums is a consequence of the fact that the Kleisli category on a category with sums has sums as well. If $+, \mathrm{in}_l, \mathrm{in}_r$ are the elements defining the sum of T_1, T_2 in $\mathcal{TS}$ then the sum of T_1, T_2 in $K_{\mathbf{Sm}}$ is defined by

$$\begin{aligned} \mathrm{in}_l^k &\stackrel{\mathrm{def}}{=} \eta(T_1+T_2)\circ \mathrm{in}_l \\ \mathrm{in}_r^k &\stackrel{\mathrm{def}}{=} \eta(T_1+T_2)\circ \mathrm{in}_r \\ f +_k g &\stackrel{\mathrm{def}}{=} f+g \end{aligned}$$

Let us construct the weak product in the binary case. Given T_1, T_2 we want to find an object $T_1 \stackrel{\diamond}{\times} T_2$, two arrows $\pi_i^k \in K_{\mathbf{Sm}}(T_1 \stackrel{\diamond}{\times} T_2, T_i)$ and a pairing map $< -, - >_k$ which maps any pair of arrows $f \in K_{\mathbf{Sm}}(T, T_1)$, $g \in K_{\mathbf{Sm}}(T, T_2)$ to an arrow $< f, g >_k \in K_{\mathbf{Sm}}(T, T_1 \stackrel{\diamond}{\times} T_2)$ such that these elements satisfy the desired equalities. Since $\mathcal{TS}$ has products and T_1, T_2 are transition systems, we know that we have in $\mathcal{TS}$ an object $T_1 \times T_2$, two arrows projections $\pi_i \in \mathcal{TS}(T_1 \times T_2, T_i)$ and the pairing map $< -, - >$. Now consider the following map $l : \mathbf{Sm}(T_1) \times \mathbf{Sm}(T_2) \to \mathbf{Sm}(T_1 \times T_2)$ defined by $l(V_1, V_2) = V_1 \times V_2$ for $V_i \subseteq S_i$. It is easily shown that l is a transition systems map. Then we define

$$\begin{aligned} T_1 \stackrel{\diamond}{\times} T_2 &\stackrel{\mathrm{def}}{=} T_1 \times T_2 \\ \pi_1^k &\stackrel{\mathrm{def}}{=} \eta(T_1)\circ \pi_1 \\ \pi_2^k &\stackrel{\mathrm{def}}{=} \eta(T_2)\circ \pi_2 \\ < f, g >_k &\stackrel{\mathrm{def}}{=} l \circ < f, g > \end{aligned}$$

It is a routine exercise to verify that this defines a weak product ∎

To be equivalent by simulation is a property which commutes to weak product as shown by the following proposition.

Proposition 8 *Let T, T_1, T_2 be three transition systems. If T is equivalent by simulation to T_1, T_2 then T is equivalent by simulation to $T_1 \stackrel{\diamond}{\times} T_2$.*

Proof: Let (for $i = 1, 2$) $\tau_i : T_i \to \mathbf{Sm}(T), \sigma_i : T \to \mathbf{Sm}(T_i)$ be the simulations given by hypothesis. Then define $\tau : T_1 \times T_2 \to \mathbf{Sm}(T)$ as the map $\tau(s_1, s_2) \stackrel{\mathrm{def}}{=} \tau(s_1) \bigcup \tau(s_2)$; τ is a map in $\mathcal{TS}$; in order to prove this suppose that $(s_1, s_2) \xrightarrow{x} (s_1', s_2') \in T_1 \times T_2$ and $s \in \tau(s_1, s_2)$; w.l.g. we can assume $s \in \tau_1(s_1)$; now $s_{11} \xrightarrow{x} s_{12} \in T_1$ and $\tau_1(s_1) \xrightarrow{x} \tau_1(s_1') \in \mathbf{Sm}(T_1)$ hence there exists $s' \in \tau_1(s_1)$ such that $s_1 \xrightarrow{x} s_2 \in T$.

[3] A category $\mathcal{C}$ has weak (small) products if for any indexed sequence of object $(C_i)_{i\in I}$ in $\mathcal{C}$ and for any indexed sequence of arrows $(f_i : D \to C_i)_{i\in I}$ there exists an object $\times_{i\in I}(C_i)$ in $\mathcal{C}$ and arrows $(\pi_i : \times_{i\in I}(C_i) \to C_i)_{i\in I}, < (f_i)_{i\in I} >: D \to \times_{i\in I}(C_i)$ such that $\pi_i \circ f = f_i$ and $< (\pi_i)_{i\in I} >= 1_{\times_{i\in I}(C_i)}$. The categorical (small) product is a weak (small) product in which the projections are natural transformations.

This proves that $\tau, < \sigma_1, \sigma_2 >$ gives the desired equivalence by simulation between T and $T_1 \overset{\diamond}{\times} T_2$.

∎

It is well known that the Kleisli category of a monad is equivalent to the category of free algebras for the same monad; it is natural to ask the question which category of algebras has the category of simulation as the subcategory of free algebras (in categorical terms we want to define the Eilenberg Moore category[4] for the monad $(\mathbf{Sm}, \eta, \mu)$).

An **Sm** *algebra* is a transition system $(S_{\mathbf{Sm}}, T_{\mathbf{Sm}})$ the set of states $S_{\mathbf{Sm}}$ of which is a poset in which any not empty subset U has a sup $\bigvee U$, and such that for $U, U' \subseteq S_{\mathbf{Sm}}$ if for all $s \in U$ there exists $s' \in U'$ such that $s \xrightarrow{x} s' \in T_{\mathbf{Sm}}$ then $\bigvee U \xrightarrow{x} \bigvee U' \in T_{\mathbf{Sm}}$.

A *morphism* between two **Sm** algebras $T_{\mathbf{Sm}}, T'_{\mathbf{Sm}}$ is a transition system map which preserves the existing sups.

5 Action algebras and Bisimulation

A *subalgebra* A' of an action algebra (A, α) is given by a subset of elements of A which is closed under the operations. By using the isomorphism between A and $\wp(At(A))$, we can consider set theoretic operations on atoms of A; hence we define a subalgebra of (A, α) as a subset A' of elements of A such that: for any $v \in V \subseteq A'$ and for any $x \in X$ the elements $\emptyset, A, \bigcup V, \bigcap V, \neg v, \alpha(x, v)$ are in A'.

We can prove then:

Theorem 1 *Two transition systems T, T' are in bisimulation iff $\mathbf{Ac}(T), \mathbf{Ac}(T')$ have an isomorphic subalgebra.*

The subalgebras of a given algebra are closed under arbitrary intersections; in particular the intersection of all subalgebras of A is (as we shall see) a subalgebra which is the smallest (w.r.t. inclusion) subalgebra of A. This minimal subalgebra has a very interesting property:

Theorem 2 *Let T be a transition system and let A_0 be the minimal subalgebras of $\mathbf{Ac}(T)$. Then the smallest transition system (w.r.t. number of states) which is in bisimulation with T is the transition system $\mathbf{Ts}(A_0)$*

The goal of this section is to prove theorems 1, 2; the key notion in these proofs is the following:

A *strong morphism* between two algebras A and A' is a one to one morphism satisfying the following equality:

$$\phi(x.v) = x.\phi(v)$$

The following lemma is a well known result [Ar].

[4] Given a monad$(\mathbf{F}, \eta, \mu)$ in a category $\mathcal{C}$ the objects of the Eilenberg Moore category $\mathcal{C}^F$ are pairs (C, γ) such that $\gamma \in \mathcal{C}(\mathbf{F}(C), C)$ satisfies $\gamma \circ \eta(C) = 1_C$ and $\gamma \circ \mathbf{F}(\gamma) = \gamma \circ \mu(C)$. An arrow $H : (C, \gamma) \rightarrow (C', \gamma')$ is an arrow $H \in \mathcal{C}(C, C')$ such that $H \circ \gamma = \gamma' \circ \mathbf{F}(H)$.

Lemma 5 *Let $f : T \to T'$ be a transition system map which is onto; then f^{-1} is a strong morphism iff for all transition $f(s) \xrightarrow{x} s' \in T'$ there exists a $s'' \in S$ such that $f(s'') = s'$ and $s \xrightarrow{x} s'' \in T$.*

Theorem 3 *Two transition systems T_1, T_2 are equivalent by bisimulation iff there exists an algebra A and two strong morphisms $\phi_1 : A \to \mathbf{Ac}(T_1)$ and $\phi_2 : A \to \mathbf{Ac}(T_2)$.*

Proof: ($\Rightarrow$) Let $T = \mathbf{Ts}(A)$ and f_1, f_2 be the transistion system maps such that ${f_1}^{-1} = \phi_1 : A \to \mathbf{Ac}(T_1)$ and ${f_2}^{-1} = \phi_2 : A \to \mathbf{Ac}(T_2)$.

Let's consider the maps

$$h_1 : T_1 \to \wp(T_2), \ h_2 \ : T_2 \to \wp(T_1)$$

defined by: $h_1(s_1) = \phi_2\{f_1(s_1)\}$, $h_2(s_2) = \phi_1\{f_2(s_2)\}$.

Let's show that h_i (for $i \in \{1,2\}$) is an arrow in $\mathcal{TS}_{\mathbf{Sm}}$ (i.e. a simulation) :

- $h_i(s) \neq \emptyset$ since ϕ_j is one to one; it follows that f_j is onto.
- Let $s \xrightarrow{x} s' \in T_i$ and $s_1 \in h_i(s)$ (i.e. $f_j(s_1) = f_i(s)$) ; we are looking for a s'_1 such that $f_j(s'_1) = f_i(s')$. Now $s \xrightarrow{x} s' \in T_i$ implies $f_i(s) \xrightarrow{x} f_i(s') \in T$; so by using lemma 5 we can find a $s'' \in S_j$ such that $s_1 \xrightarrow{x} s'' \in T_j$. The result follows then by putting $s'_1 = s''$.

It follows that we have two simulations $h_1 \ : T_1 \to \mathbf{Sm}(T_2), h_2 \ : T_2 \to \mathbf{Sm}(T_1)$. In order to prove that T_1 and T_2 are equivalent by bisimulation, it's enough to show that the relations induced by h_1, h_2 are mutually inverses.

For this it is sufficient to prove that for all $s \in S_1, s' \in S_2$:

$$s' \in h_1(s) \iff s \in h_2(s')$$

And this follows from the following identities:

$$\begin{aligned} s' \in h_1(s) &\iff f_2(s') \in \{f_1(s)\} \\ &\iff f_2(s') = f_1(s) \\ &\iff f_1(s) \in \{f_2(s')\} \\ &\iff s \in h_2(s') \end{aligned}$$

($\Rightarrow$)

Let's define $\mathcal{B}_* \stackrel{\text{def}}{=} \mathcal{B} \bigcup \mathcal{B}^{-1}$ where $\mathcal{B}$ is an arbitrary bisimulation between T_1, T_2.

Let's then define an equivalence relation $s \sim s'$ on elements of $S_1 \bigcup S_2$ as follows:

$s \sim s'$ if and only if

$$\exists s_1, \ldots, s_n (s = s_1, \ s' = s_n, \ (s_1, s_2) \in \mathcal{B}_*, \ldots (s_{i-1}, s_i) \in \mathcal{B}_*, \ldots, (s_{n-1}, s_n) \in \mathcal{B}_*$$

Let's now define a transistion system T whose set of states is $S = (S_1 \bigcup S_2)/\sim$ and whose transistions are defined by:

$[s_1] \xrightarrow{x} [s_2] \in T$ if and only if for all $s \in [s_1]$ there exists a $s' \in [s_2]$ such that $s \xrightarrow{x} s' \in T_1 + T_2$.

We now show that $s_i \xrightarrow{x} s'_i \in T_i$ implies $[s_i] \xrightarrow{x} [s'_i] \in T$ (this means that for all s such that $s \sim s_i$ there exists a s' such that $s' \sim s'_i$ and $s \xrightarrow{x} s' \in T_1$ or $s \xrightarrow{x} s' \in T_2$).

This property follows from the following diagram ($s\mathcal{B}_* s'$ is an abbreviation for $(s, s') \in \mathcal{B}_*$)

$$\begin{array}{cccccccc} s_i & \mathcal{B}_* & s_1 & \mathcal{B}_* & \cdots & s_n & \mathcal{B}_* & s \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ s'_i & \mathcal{B}_* & s'_1 & \mathcal{B}_* & \cdots & s'_n & \mathcal{B}_* & s' \end{array}$$

We deduce then that the maps $\iota_1 \ : s_1 \mapsto [s_1]$ and $\iota_2 \ : s_2 \mapsto [s_2]$ are transition system maps ($\iota_1 \ : S_1 \to S$, $\iota_2 \ : S_2 \to S$) which are onto.

In order to prove that their inverse images are strong morphisms we are left to prove for $i = 1, 2$, $\iota_i^{-1}(x.V) = x.(\iota_i^{-1}(V))$: now if $[s_i] \xrightarrow{x} [s] \in T$ and $s_i \in S_i$ then, as ι_i is a transistion system map, we find a $s' \in [s], s'_i \in S_i$ such that $s_i \xrightarrow{x} s' \in T_i$. This proves that ι_i is a transition system map which satisfies the hypothesis of lemma 5, and by consequent ι_i^{-1} is a strong morphism. ∎

Lemma 6 *Let A be an algebra*

- *Let $\mathcal{A}$ be a family of subalgebras of A; then $\bigcap \mathcal{A}$ is a subalgebra of A. In particular there is a* minimal subalgebra *of A*
- *If $\phi : A \to A'$ is a strong morphism, then $\phi(A)$ is a subalgebra of A'.*

Proof: Trivial. ∎

Let (A', α') be a subalgebra of (A, α) and consider the following equivalence relation on $\mathrm{At}(A)$:

$$a \simeq a' \text{ iff } \forall v \in A' (a \in v \iff a' \in v)$$

We will write the equivalence class of a as $[a]$.

Lemma 7 *For any $a \in \mathrm{At}(A)$ the element $[a] \in A'$*

Proof: Let $v \stackrel{\text{def}}{=} \bigcap \{v' \in A' | a \in v'\}$ ($v \neq \emptyset$ because $a \in 1 \in A'$); assume that there exists $b \in v - [a]$; since $b \notin [a]$ there exists $v_1 \in A'$ such that $b \in v_1, a \notin v_1$; hence $a \in v - v_1, b \notin v - v_1$ and since $v \subseteq v - v_1$ we conclude $b \notin v$, which is absurd; hence $v = [a]$. ∎

Now we can prove the following proposition:

Proposition 9 *Let A, A_1 be algebras and A' a subalgebra of A. The following statements are true:*

- *A' is atomic.*
- *A strong morphism $\phi : A_1 \to A$ is an isomorphism between A_1 and $\phi(A_1)$.*

Proof:

- It is easily seen that the atoms in A' are the classes $[a]$ for $a \in A$.

- We will show that $[b]_{\phi(A_1)} = \phi(\phi_*(b))$. We have $b' \in \phi(\phi_*(b))$ iff $\phi_*(b') = \phi_*(b)$, hence for any $u \in A_1$ ($b \in \phi(u) \iff b' \in \phi(u)$) which proves equivalence of b, b'.

  On the other hand if $b' \simeq b$ then $b' \in \phi(\phi_*(b))$ because $b \in \phi(\phi_*(b))$.

■

Proof of theorem 1: Immediate by theorem 3 and proposition 9.

Proof of theorem 2: As A_0 is a subalgebra of $\mathbf{Ac}(T)$, from theorem 1 we deduce that $T_0 = \mathbf{Ts}(A_0)$ and T are equivalent by bisimulation. Now if T' is in bisimulation with T then it is in bisimulation with T_0 too (again by theorem 1).

Now the number of states in T' is the cardinal of the set of atoms in $\mathbf{Ac}(T')$ which is greater then the cardinal of the set of atoms in $\mathbf{Ac}(T_0) = A_0$ which is the number of states of T_0.

To end this section by showing in an example the notion of algebra and minimal subalgebra at work:

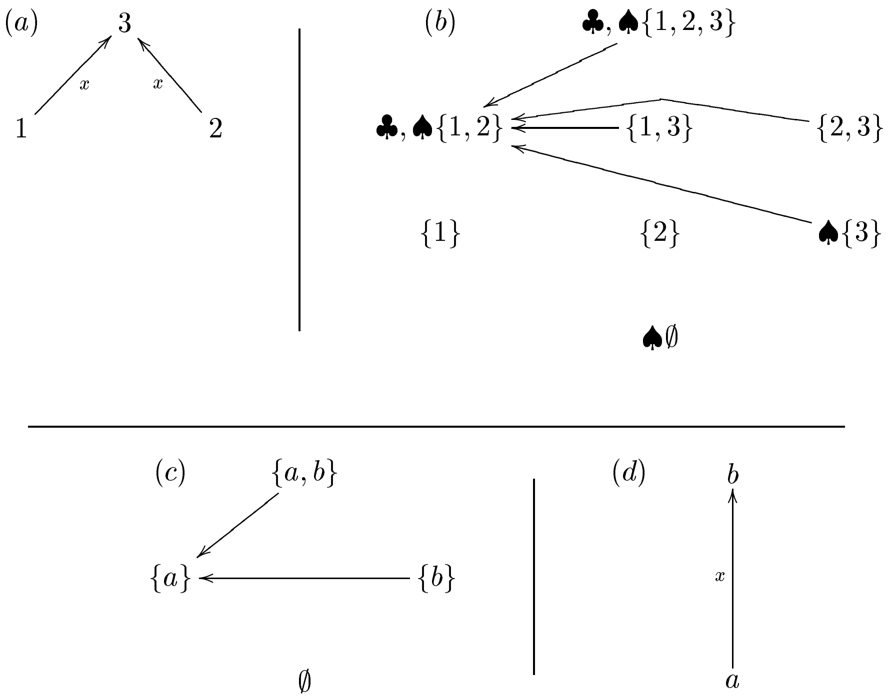


fig 1

In (a) a transition system is shown, in (b) we can see the associated algebra (the arrows represent the actions; the elements with a ♠ indicates the elements of the minimal subalgebra).

(c) shows the algebra corresponding to the minimal subalgebra of (b) and (d) shows the transition system which corresponds, by duality, to the algebra of (c). This transition system is the smallest element of the class of transition systems in bisimulation with the system in (a).

6 Skeleton of an action algebra

Note that in the case of a CBA the notion of minimal subalgebra is trivial, the latter always being the algebra $\{0, 1\}$.

The presence of actions in the category $\mathcal{AL}$ makes this notion not trivial since for any $x \in X$ the element $x.1$ (which in general is neither 1 nor 0) must be in the minimal algebra. Hence we are looking for a set Σ_A, the *skeleton* of the algebra A that is the smallest subset of A containing 1 and closed under linear actions.

Formally, let A be an algebra and consider the following sequence of sets:

- $A_0 = \{1\}$
- $A^{n+1} = \{x.v \mid v \in A^n,\ x \in X\}$

Then the skeleton of A is

$$\Sigma_A = \bigcup_{n \in \omega} A^n$$

An immediate consequence of the definition is that Σ_A is included in the minimal subalgebra of A.

Moreover the skeleton has a natural structure of transition system: the states of Σ_A are the sets of states of the shape $x_1. \cdots .x_n.1 \neq \emptyset$ (for all $x_1, \ldots, x_n \in X$) and a transition $V \xrightarrow{x} V'$ is in Σ_A if and only if $V' = x.V$. (Note that indeed Σ_A is a deterministic automaton, 1 being the inital state and all elements being final states).

We define then a *skeleton homomorphism* between two skeletons Σ, Σ' as a transition systems morphism which preserve the root (i.e. the 1 of the algebra) and investigate the equivalence induced by skeleton isomorphism (i.e. T and T' are *skeleton equivalent* if and only if they have isomorphic skeleton) which we note $\simeq_\Sigma$.

This is a rather weak equivalence as shown in the following picture presenting two transition systems with isomorphic skeletons but which are not in bisimulation ($\clubsuit$ denotes the elements of the skeleton):

$$T_1 \quad \begin{tikzcd} 0 \arrow[loop left, "x"] \arrow[rr, "x"] & & 1 \end{tikzcd} \qquad \mathbf{Ac}(T_1) \quad \begin{tikzcd} & \clubsuit\{0,1\} \arrow[dl] & \\ \clubsuit\{0\} & & \{1\} \arrow[ll] \\ & \emptyset & \end{tikzcd}$$

$$T_2 \quad \begin{tikzcd} 0 \arrow[loop left, "x"] & & 1 \end{tikzcd} \qquad \mathbf{Ac}(T_2) \quad \begin{tikzcd} & \clubsuit\{0,1\} \arrow[dl] & \\ \clubsuit\{0\} & & \{1\} \\ & \emptyset & \end{tikzcd}$$

It is now time to recall two possible equivalences between transition systems:

- We say that two transition systems T, T' have the same language if for any word $w = x_1 \ldots x_n$ there exists a path labelled by w in T if and only if there exists a path labelled by w in T'.
- T and T' are trace equivalents if and only if there exists a relation $R \subseteq S \times S'$ such that:

  1 For all $s \in S$ there exists an $s' \in S'$ such that $(s, s') \in R$.

  2 For all $s' \in S'$ there exists an $s \in S$ such that $(s, s') \in R$.

  3 For any couple $(s, s') \in R$, s and s' produce the same language (i.e. for any word $w = x_1 \ldots x_n$ there exists a path with origin s labelled by w in T if and only if there exists a path with origin s' labelled by w in T').

We will prove now that the skeleton equivalence is included in language equality and includes trace equivalence.

The first inclusion is immediately deduced by the following proposition

Proposition 10 *Let T, T' be two transition systems and $f : \Sigma \to \Sigma'$ a skeleton morphism. Then any finite word produced by T is produced by T' as well.*

Proof: Let $w = x_1 \cdots x_n$ be a finite word produced by T; this means that there exist $s_1, \ldots, s_{n+1}$ in S and a path of transitions $s_1 \xrightarrow{x_1} s_2 \xrightarrow{x_2} \cdots s_n \xrightarrow{x_n} s_{n+1}$ in T.

We state that this is the case iff the set $x_1 . \cdots . x_n . 1 \neq \emptyset$.

Indeed if the path $s_1 \xrightarrow{x_1} s_2 \xrightarrow{x_2} \cdots s_n \xrightarrow{x_n} s_{n+1}$ is in T then $s_n \in x_n.1$ hence $s_{n-1} \in x_{n-1}.s_n \subseteq x_n.1$ and so on till we have $s_1 \in x_1 . \cdots . x_n . 1$.

On the other hand pick an element $s_1 \in x_1.\cdots.x_n.1 \neq \emptyset$, then by definition s_1 has a predecessor for an x_1 transition by an element $s_2 \in x_2.\cdots.x_n.1 \neq \emptyset$, and so on till we can find an element $s_{n+1} \in 1$ which gives us the final state of the path. Hence if f is a morphism from the skeleton of T to the skeleton of T' the word w is a word in T' as well. ∎

The other inclusion is given by the following proposition

Proposition 11 *Let T and T' be two trace-equivalent transition systems : then $T \simeq_\Sigma T'$.*

Proof: Given a word $w = x_1 \cdots x_n$ we will note the element of the skeleton which corresponds to $x_1.\cdots.x_n.1$ as $w.1$. Moreover 1 and $1'$ will denote respectively the maximal elements in $\mathbf{Ac}(T)$ and $\mathbf{Ac}(T')$.

We will show that if T and T' are trace equivalents, then for all word w, w' on the alphabet X, we have :

$w.1 = w'.1 \neq \emptyset$ if and only if $w.1' = w'.1' \neq \emptyset$

Once proven this property we can end the proof since the map which associate $w.1'$ to $w.1$ is a skeleton isomorphism.

Let's assume that $w.1 = w'.1 \neq \emptyset$ and let $s \in w.1$. By the definition of skeleton, s is the origin of a path labelled by w if and only if s is the origin of a path labelled by w'.

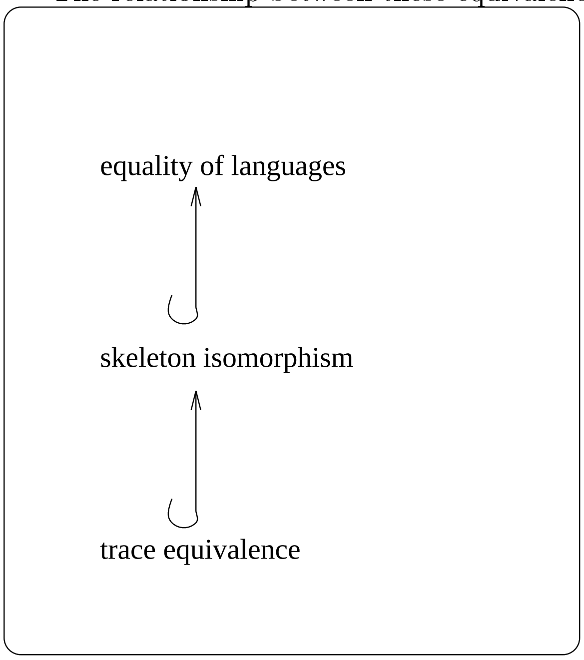
It follows that any state $s' \in w.1'$ such that $(s, s') \in \mathcal{R}$ is in $w'.1'$.

Let's assume that there exists an $s' \in w.1'$ such that $s' \notin w'.1'$. Then for any s such that $(s, s') \in \mathcal{R}$ we have $s \notin w'.1$: contradiction.

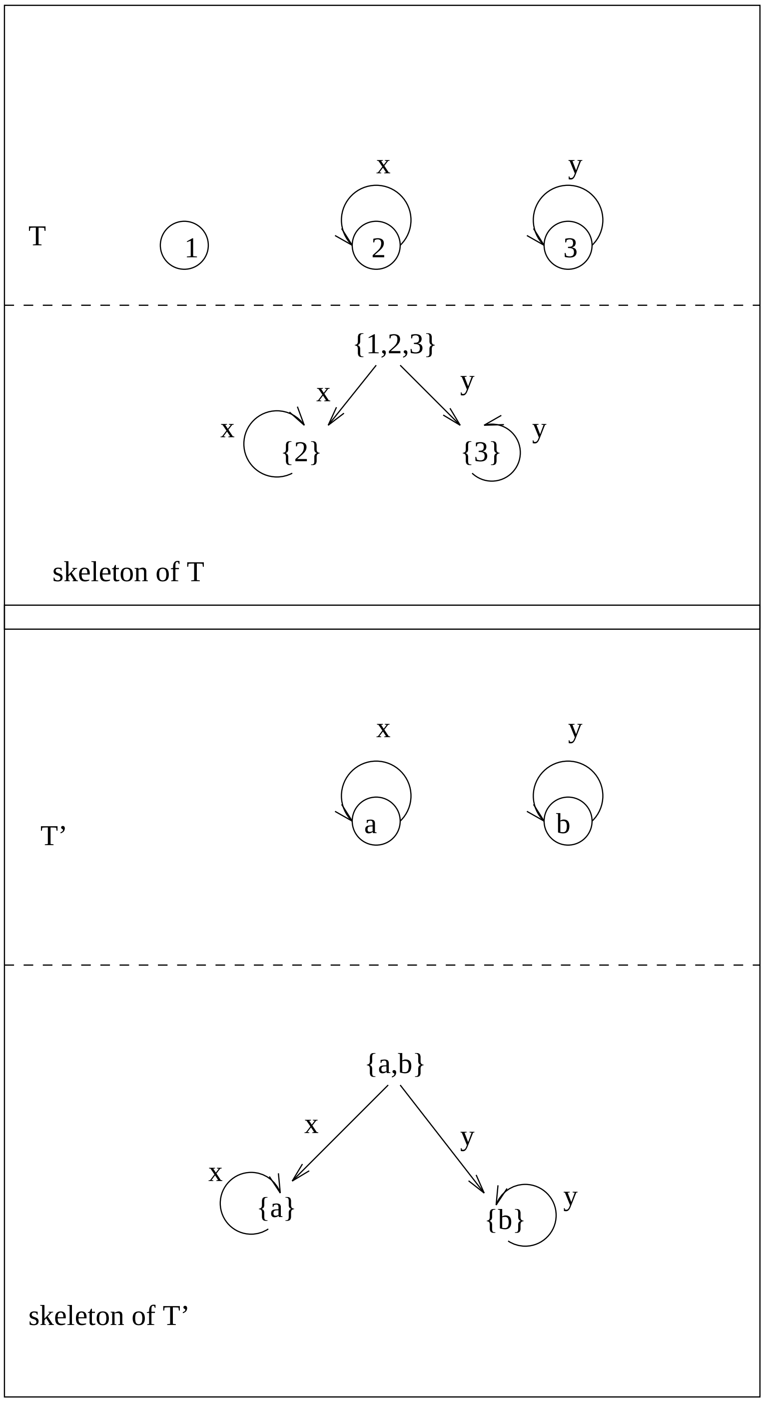
We deduce that $w.1' = w'.1' \neq \emptyset$.

Since the argument is symmetrical if follows $w.1 = w'.1 \neq \emptyset$ if and only if $w.1' = w'.1' \neq \emptyset$. ∎

The relationship between these equivalences can hence summarized as follows:



These inclusions are strict. The following picture shows two systems which have isomorphic skeleton but are not trace equivalents



The following picture shows two transition systems which have the same language but whose skeletons are not isomorphic :

$$T_1 \quad \begin{tikzcd} 0 \arrow[loop left, "x"] \arrow[r, "x"] & 1 \arrow[r, "x"] & 2 \end{tikzcd} \quad \text{whose skeleton is} \quad \begin{tikzcd} \{0,1,2\} \arrow[d] \\ \{0,1\} \arrow[d] \\ \{0\} \arrow[loop left] \end{tikzcd}$$

$$T_2 \quad \begin{tikzcd} 0 \arrow[loop left, "x"] \arrow[r, "x"] & 1 \end{tikzcd} \quad \text{whose skeleton is} \quad \begin{tikzcd} \{0,1\} \arrow[d] \\ \{0\} \arrow[loop left] \end{tikzcd}$$

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