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Strong normalization for simply typed λ calculus with pairings and projections

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1. Introduction

We propose a new, elegant and self-contained proof of simply typed λ calculus with pairings and projections.

Definition 1.1. (The grammar)

$$t ::= x|\lambda x.t|\langle u, t\rangle|\pi_0 t|\pi_1 t|(u)t$$

for simplicity we denote $t_1 t_2 ... t_n = (...(t_1)t_2)...t_{n-1})t_n$. We adopt Krivine's notationfor parentheses. Thus in $\pi^*(u)t$, projections are applied to the result of the application of u to t.

We remind that every term of the λ-calculus with pairings and projections has the form

$$\lambda y_1 ... \lambda y_m.(\pi^{*n}(...(\pi^{*1} ht)t_1 ...)t_n$$

where π^{*i} is a sequence, possibly void, of projections π_0 or π_1 and h, the *head*, has one of the following forms:

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$$h := x|\lambda x.t|\langle t_0, t_1\rangle$$

We define SN to be the set of *strong normalizable terms* and $\succ^*$ the reflexive and transitive closure of the β reduction $\succ$.

Definition 1.2. A lambda term t is *elementary* if $t \notin SN$ but all the proper subterm of t are in SN.

Note that any elementary term is of the form

- $(\pi^{*n}(...(\pi^{*1}(\lambda x.u)t)t_1...)t_n$ (with $u, t, t_1, ..., t_n \in SN$);
- $(\pi^{*n}(...(\pi^{*1}\pi_i\langle u_0, u_1\rangle)t_1...)t_n$ (with $i = 0$ or $i = 1$ and $u_0, u_1, t_1, ..., t_n \in SN$).

where $(\lambda x.u)t$ or $\pi_i\langle u_0, u_1\rangle$ is the *head redex*. For example $(\lambda x.xx)\lambda x.xx$ or $(\pi_0(\pi_1\langle z, \lambda y.\langle\lambda x.xx, w\rangle\rangle)y)\lambda x.xx$ are elementary terms. Note that the head of an elementary term must reduce otherwise, as $t_1, ..., t_n \in SN$, it would surely not have an infinite reduction.

Proposition 1.1. If $v \notin SN$ then v has an elementary subterm.

Definition 1.3. Given $t \notin SN$ we define:

- the *standard subterm* of t as its leftmost elementary subterm;
- the *standard redex* of t the head redex of its standard subterm;
- the β *standard reduct* of t the term t' obtained from t contracting the standard redex. We will write $t \succ_{st} t'$;
- the *standard reduction* of t the succession of terms obtained from t applying only β standard reductions. Thus if $t \succ_{st} t_1 ... \succ_{st} t_n \succ_{st} ...$ then the standard reduction of t is $t, t_1, ..., t_n, ...$.

Proposition 1.2.

1. if $(\pi^{*n}(...(\pi^{*1}u[t/x])t_1...)t_n \in SN$ then $(\pi^{*n}(...(\pi^{*1}(\lambda x.u)t)t_1...)t_n \in SN$;
2. if $(\pi^{*n}(...(\pi^{*1}u_i)t_1...)t_n \in SN$ and $u_{1-i} \in SN$ then $(\pi^{*n}(...(\pi^{*1}\pi_i\langle u_0, u_1\rangle)t_1...)t_n \in SN$.

Proposition 1.3. $t \notin SN$ iff the standard reduction of t is infinite.

Proposition 1.4. Let $u, t \in SN$ and $u[t/x] \notin SN$ then there exists v such that $u[t/x] \succ^*_{st} v$ and the standard subterm of v is in the form

$$(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$$

Proof:
by in induction on $h(u)$ where, given a term s, $h(s)$ is the length of the maximal reduction of s. Let consider the possible standard reductions that $u[t/x]$ can perform:

- $u[t/x] \succ_{st} u'[t/x]$ apply inductive hypothesis on u';

- $u[t/x] \succ_{st} u'$ because of a redex in t then, as the standard reduction reduces the head redex of the standard subterm, the standard subterm of $u[t/x]$ is $(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$ and we end with $u[t/x] = v$;
- $u[t/x] \succ_{st}^{*} u'$ and the redex is not in t nor in u. Therefore t is a lambda abstraction or a pair in order for $u[t/x]$ to reduce:
    - t is a lambda abstraction and the standard subterm of $u[t/x]$ is in the form $(\pi^{*n}(...(\pi^{*2}(t)t_1)t_2...)t_n$. We end with $u[t/x] = v$;
    - t is a pair and the standard subterm of $u[t/x]$ is in the form $(\pi^{*n}(...(\pi^{*1}\pi_i t)t_1...)t_n$. We end with $u[t/x] = v$.

□

Proposition 1.5. Let ut be elementary. There exists an elementary term $(\lambda x.s)t$ such that $ut \succ_{st}^{*} (\lambda x.s)t$.

Proof:
by induction on $h(u)$. We remind that, by definition 1.2, the head of an elementary term must be a redex. Therefore the cases considered are the following:

- $ut = (\lambda x.s)t$ and we have the result;
- $(\pi^{*n}(...(\pi^{*1}(\lambda x.v)t')t_1...t_{n-1})t \succ (\pi^{*n}(...(\pi^{*1}v[t'/x])t_1...t_{n-1})t$. Apply inductive hypothesis with $u = \pi^{*n}(...(\pi^{*1}v[t'/x])t_1...)t_{n-1}$;
- $(\pi^{*n}(...(\pi^{*1}\pi_i\langle u_0, u_1\rangle)t_1...t_{n-1})t \succ (\pi^{*n}(...(\pi^{*1}u_i)t_1...t_{n-1})t$. Apply inductive hypothesis with $u = \pi^{*n}(...(\pi^{*1}u_i)t_1...t_{n-1}$.

□

Proposition 1.6. Let ut be elementary. There exists v such that $ut \succ_{st}^{*} v$ and the standard subterm of v has the form $(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$ and the head t is obtained by substitution of t to some x in u.

Proof:
by proposition 1.5 and 1.4. □

Theorem 1.1. Every simply typed lambda term z with pairing and projection is strong normalizable.

Proof:
suppose $z \notin SN$. Let E be the set of typed elementary terms: E is not empty. Consider some $ut \in E$ such that the length (logical complexity) of the type of t is minimal among all the $ut \in E$ and all the type assignments given to such ut. Then for no $u't' \in E$ there is an assignment of a shorter type to t'. By proposition 1.6 $ut \succ_{st}^{*} v$ and the standard subterm of v has the form $(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$. The type of t has the same type it has in ut by subject reduction because ut reduces to $(\lambda x.t')t$, then to $t'[t/x]$ and one substitution of t to x produces the head of $(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$. The type of t strictly includes the type of t_n. Now $(\pi^{*n}(...(\pi^{*1}t)t_1...)t_n$ is a term in E of the form $u't_n$ with an assignment to t_n of a shorter type than t but this contradicts the choice of the type assignment for t. Therefore $E = \emptyset$ and $z \in SN$ because it can not have any elementary subterm. □