THE PHYSICS BEHIND THE SODACONSTRUCTOR

by Jeckyll

CONTENTS

1. INTRODUCTION

The sodaconstructor applet is simply a simulator of the mechanical laws of physics. In the following chapters of this article, we will investigate these physical laws and the way they have been implemented in the applet. Here I wish to express just a few preliminary considerations.

First, I wish to say that in this work you will not find anything really original in the scientific meaning of the term. Everything reported in this paper is a well-known notion of physics and mathematics. Therefore, this article can't be catalogued in the category of scientific work, but rather in the category of didactical publication. Thanks to this paper, anyone with a minimal knowledge of the main mathematical rules of analysis can discover with his own hands how the sodaconstructor applet works.

A second thing that I wish to say here is particularly addressed to anyone who because of his young age or particular field of study doesn't totally understand the mathematics in this work. For you I repeat another time that there is nothing special presented in this work. All these things, scientifically speaking, are trivial. If you wish to see something really original, then look for models in the sodazoo. There you will find genuine creativity. Let me use a simple analogy that I think better explains what I'm trying to say. The sodaconstructor applet is something like a musical instrument. It isn't particular important who made a piano; it is more important who plays the piano! I'm not sure of this, but I would bet money that Mozart and Beethoven knew nothing about the mechanism behind their own pianos. So, I wish to stress one thing: please, don't stop playing the piano.

Finally, I wish to provide here a short description of the contents of this work. This paper is organized in chapters, further separated, where necessary, into subsections. In the second chapter particular units of measurement are studied. In the third chapter virtual experiments are realized in order to investigate the physical constants adopted by the applet. In this chapter the functionalities of the applet's cursors of gravity, friction and stiffness are experimentally determined. All these experiments have been adequately described and the corresponding links are included in the text of the explanations. In the fourth chapter, the main cinematic quantities for the description of the motion of free masses are introduced and discussed. In the fifth chapter, a simple definition of a model is introduced. The sixth chapter gives a detailed

analysis of the forces implemented in the applet. The seventh chapter provides a detailed mathematical explanation of the behavior of springs and muscles. In this chapter the functionalities of the cursors of the muscle control panel are completely clarified. In the eighth chapter, the equations of motion for a generic model are formulated, while in the ninth chapter, we find a simple numerical procedure for their integration. In the tenth chapter, a simplified method for the study of models is introduced. This chapter studies polygons, tension springs, and linear motors. Finally, the eleventh chapter contains the conclusion and acknowledgements. The paper is also accompanied by three appendixes, in which specific mathematical arguments are studied in-depth.

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2. UNITS OF MEASUREMENT

In the real world, especially in scientific and technical communications, all physical quantities are expressed in units of measurement defined by the International System (S.I.). We all know that the unit of measurement for length is the meter (*m*), the unit of measurement for time is the second (*s*), the unit of measurement for mass is the kilogram (*kg*), and so on for the other fundamental quantities (temperature, intensity of current, etc.). In the soda universe, however, we can't use these units of measurement. To illustrate why, let's explore this seemingly simple question: how long, for example, is the old charming daintywalker? I have a 17-inch monitor, and I measure a length of 7.9 *cm*. Is this the right answer? Certainly not. Another person with a different size monitor would find a different measure. Furthermore, it is even possible that two different people, each with the same size monitor, would find two different measures because their monitors are of different brands. It's clear why it isn't adequate to use the meter (and its submultiples) as the unit of measurement for length in the sodaconstructor: the measurement would be slightly different for each person.

Obviously, in the soda universe it is essential to choose another unit of measurement for length. Since the amount of pixels in the sodaconstructor window is the same for all users ($657\ pxl \times 428\ pxl$: see Dimensions of the Sodaconstructor Window), independent of the dimensions of the monitor screen, in this paper we will use the pixel as the unit of measurement for length. The symbol that we will adopt will be: *pxl*. The main problem with this new unit of measurement is this: how is it possible to measure something in pixels without counting all the pixels one by one? This is not a trivial question, because to count many pixels on the screen is excruciatingly painful. Therefore, I've used the following technique: I know that a fixed mass (I really don't like this denomination, I would have preferred fixed point) has the following dimensions: $6\ pxl \times 6\ pxl$. So, by arranging many fixed masses like a chessboard, it is possible to measure the distance between two different points on the screen by simply counting the fixed masses between the two points and multiplying the amount by 6. Obviously, it is possible that the length we want to measure is not a multiple of 6; in this case we will arrange the fixed masses like a chessboard until we are less than 6 pixels from the endpoint, and we will count the remaining pixels, 5 at the most. Sometimes, however, the length that we want to measure is so great that the above *fixed masses method* would be equally painful for our eyes and patience. In this case it is possible

to apply a mixed method, such as using fixed masses to measure distances greater than 6 pixels. One of my favorite ways to get equal distances between fixed points is by constructing springs at 0°, 45° and 90°, angles which are easy to precisely create in the sodaconstructor. Using these lines to make perfect squares, it is easy to replicate a fixed length many times, in order to measure using larger increments. I've used this technique often, as you will see in examples referred to later (you will find one example in Dimensions of the Sodaconstructor Window).

Another physical quantity involved in these studies is time. Again, it is impossible to use the usual unit of measurement (*seconds s*). In principle it should be possible to use the *second* (*s*) as a measurement of time, because advanced simulation programs like our beloved Sodaconstructor should synchronize their calculus with the internal microprocessor clock. For this reason, the duration of the simulation should be uniform in all different computers. However, all computers are different from each other. Computers today are assembled with many different parts: there are the microprocessors, the main boards, the accelerated graphic devices, and many other such devilries. (Are there any of you young people old enough to remember the simplicity of the Commodore 64? I bet you don't.) The real performance of a computer depends greatly on all its different components. Therefore, it is very difficult to find two different computers with identical performances. For these reasons, I suspect that the same simulation could have different durations on different computers. It may seem that I am being unnecessarily precise. Nevertheless, as you will see in the next chapter, I will need maximum precision to get reliable results.

Just as with pixels, the number of frames necessary to complete a simulation is exactly the same for each computer. So, to avoid any ambiguity, we will use the frame as our unit of measurement for time. The symbol that we will adopt will be: *frm*. Now the problem is: how can we count the amount of frames that it takes for a particular model to complete a simulation? The surprising answer is: using an ordinary chronometer. This seems like a contradiction, but it isn't. The main problem is that in the sodaconstructor applet there isn't a frame counter, so the only thing that we can use is a chronometer. Using the new sodarace timetrial applet, each of us can find how many frames there are in a second for our own computer's performance. Those of us with a computer of high performance will get many frames in a second. Those of us with a computer of less power will not get as many frames in

a second. Thanks to the synchronicity between the simulator's computation and the internal clock of the computer, the difference between two different computers will probably never be very high. Still, I think there will always be a difference. In any case, if each of us finds for his own computer how many frames there are in a second, it solves the problem. After this has been determined, the length in frames of a particular simulation can be calculated by multiplying the number of seconds by the number of frames per second. The result in frames should be the same for everyone, completely independent of the computer's performance.

The last question is: how can I find how many frames there are in a second of my computer's work? This can be determined using any model in the sodarace timetrial. In order to make this as accurate as possible, I made a very slow model (Slow walker). In my computer it covered the whole route in 966 *sec* (16′06″: an eternity) for a total of 107036 *frm*. So, the time conversion ratio (t_{cr}) for my computer is:

$$t_{cr} = \frac{107036}{966} = 110.81 \ \frac{frm}{sec} \tag{2.1}$$

This ratio is very important because I use it in the virtual experiences described in the next chapter.

To review, we will be using the following units of measurement:

Table 2.1: The new units of measurement

Physical quantity	**Unit of measurement**
Length	*pxl (pixel)*
Time	*frm (frame)*
Velocity	pxl/frm
Acceleration	pxl/frm^2

What about mass? We know that the other fundamental physical quantity involved in the dynamic problem is mass. Well, in the 8th chapter I will show you mathematically that it isn't

necessary to assume any value for the masses. This is because all masses in the sodaconstructor applet are the same.

3. DETERMINATION OF THE PHYSICAL CONSTANTS ADOPTED BY THE APPLET

As we all know, there are many constants present in the physical laws that govern our world. One of the most important constants, one that often persecutes students in school, is the earth's gravitational acceleration constant *g*. In the proximity of the Earth's surface, the value of this constant is about $g = 9.8\ m/sec^2$. Thanks to the knowledge of this constant, we can say many things. First, we can say that the velocity of a free falling body close to the Earth's surface increases $9.8\ m/sec$ every second of its falling (this is true only until air friction begins to have an effect; i.e. just in the first few seconds of its falling). Another thing that this constant allows us to say it is that the weight *W* (expressed in Newton *N*) of a generic mass *m* (expressed in *kg*) is obtained by the rule:

$$W = m \cdot g \tag{3.1}$$

These apparently simple things allow us to see how the knowledge of the constant *g* is, without a doubt, very important for a lot of physical and engineering applications. Obviously, in addition to the constant *g*, there are a lot of other constants that are equally important. All these constants have specific values commonly known by the scientific community. The question is: how were these constants originally found? The answer is very simple: by means of experimental tests.

How does all this relate to the sodaconstructor? To answer this question we must first understand exactly what the sodaconstructor is. The sodaconstructor is simply a simulator of physical laws. In the sodaconstructor there are a number of mechanical laws[1] that, like in reality, cannot be violated. Therefore, in principle, it should be possible to determine the physical constants of the laws used in the sodaconstructor by means of virtual experiments, much like the experimental tests that allow us to understand nature in the real world.

Of course, if I were an expert in computer languages I would simply find the sodaconstructor's physical constants by looking for them in the soda algorithm. Unfortunately, I am not an expert. So the best thing that I can do is transform myself into a

[1] The Newtonian laws of dynamics, Hooke's law for the springs, the Newtonian law for the fluid's friction, the laws of the quasi-elastic impact between masses and walls.

virtual Galileo (Italian scientist of the seventeenth century) in order to investigate the soda universe.

I've created a sodaconstructor Laboratory where I've executed many virtual experiments. Thanks to these experiments I was able to find how the well-known cursors for gravity (*g*), friction (*f*) and stiffness (*k*) actually work (see Fig. 3.1).

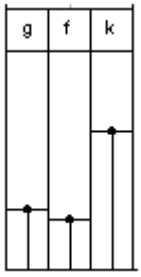

Figure 3.1: The physical sodaconstructor constants

In the following parts of this chapter I will explain exactly what I've determined about each one of the above constants. I've also investigated about the coefficients of dynamical restitutions of the walls and ground.

3.1 *THE GRAVITY*

In the real universe we could think of a simple experimental test to determine the gravitational acceleration constant *g*. We could let a little object like a stone fall from a fixed height *h*. Meanwhile we could measure the time *t* it takes the stone to fall by means of a chronometer. Knowing that the falling stone moves with a uniformly accelerated motion, we could use the following formula:

$$h = \tfrac{1}{2} g t^2 \tag{3.2}$$

to get the acceleration g:

$$g = \frac{2h}{t^2} \tag{3.3}$$

In fact, this method of determining the constant *g* isn't very accurate, because air friction will have an effect. A more accurate method of determining the constant *g* is the pendulum method. It is possible to find the constant *g* by simply measuring the period of a complete oscillation of a long pendulum[2].

What can we do in the soda universe? Exactly the same things! We could get the acceleration constant[3] *g* by using the experiment of a mass falling. Fortunately, in the soda universe we have the option to turn off the "air" friction completely, so we don't need to worry about the problem described above. Still, this method is a bit problematic. First, I don't have an accurate chronometer; I just measure time with my analog clock, which means I only have the accuracy of one second (an eternity compared to the accuracy required in this sort of experiment). Second, it is very difficult to start the chronometer exactly when the mass begins falling. Third, it is also very difficult to stop the chronometer exactly when the mass hits the ground. As a result, I would get a measure with an intolerable error. What could I do to fix this problem? I could conduct this experiment a number of times, so that I could reduce the margin of error. But this method it is too long and tedious, even for my patience. However, there is a more convenient and accurate solution: to use the pendulum method. Using as long a pendulum as possible and measuring its period[2], we can discover the gravitational acceleration constant *g*.

In Appendix A, I will explain in a detailed manner the mathematical theory behind the pendulum method. Therefore, in the following description, I will restrict myself to explaining only the main relation that we will use.

As discussed in Appendix A, when the maximum angular excursion α_{max} (expressed in radiant *rad*) of a pendulum is such that it is possible to use the following approximation:

$$\sin \alpha_{max} \approx \alpha_{max} \tag{3.4}$$

[2] The time that the pendulum needs to reach its maximum excursion starting from the identical position.

[3] There is a point I want to make clear about the word "constant" in the soda universe. As we well know it is possible to change the gravity by moving the above mentioned gravity cursor. Therefore, in principle, the gravity isn't constant. Nevertheless, if we choose a particular level of gravity, our models are moving with a value of gravity that doesn't change over time; in this case the gravity is constant.

(just when the value of α_{max} is very small; see Fig. 3.2) the period T of an oscillation depends only on the length l of the pendulum and the gravity constant g, by means of the formula:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{3.5}$$

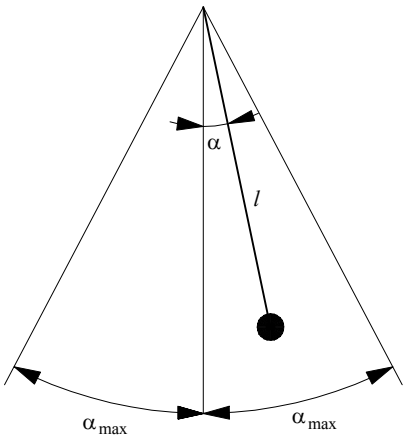

Figure 3.2: Angular excursion of a pendulum

By measuring the period T and knowing the length of the pendulum it is possible to get the constant g by means of the following inverse formula:

$$g = 4\pi^2 \frac{l}{T^2} \tag{3.6}$$

Some might argue that this unnecessarily complicated, because to measure the period T I will have the same difficulties as described for the falling mass method. This isn't true! If friction is dropped to zero, the pendulum's oscillation could last forever. Therefore, it will be possible to measure not just one complete oscillation but many oscillations. In this way the inevitable errors mentioned above will be distributed in many oscillations, reducing their effect on the final computation. This is exactly what I've done. Fixing a particular value for the gravity by

moving the well-known cursor[4], I measured the elapsed time for many oscillations. Then, I divided the overall time by the number of complete oscillations, giving me the period T of a single oscillation with the required accuracy.

Before I start explaining the virtual experiences about gravity, I will say just one thing about the cursors shown in Figure 3.1. Each one of these cursors can assume 108 different positions. Each of these 108 positions differs from the previous or the following position by one pixel of displacement. We have position 0 when the cursor is at the bottom (in this position the physical quantity associated with the cursor has the value of zero), and we have position 107 when the cursor is at the top (in this position the physical quantity associated with the cursor has the maximum value).

In the following section we will look for the rules of variation for the constants g, f, and k in respect to their cursor position. So, naming these positions p_g, p_f and p_k, we will look for the following three laws:

$$\begin{cases} g(p_g) \\ f(p_f) \\ k(p_k) \end{cases} \quad \text{with} \quad p_g,\ p_f, p_k = 0,1,2,\ldots,107 \tag{3.7}$$

• ***Experiment 1***

Using Experiment 1 it was possible to find the value of the gravitational acceleration constant g (in pxl/frm^2) when the gravity cursor position is $p_g = 20$. The length of the pendulum in this case is $l = 374\ pxl$.

I measured the time it took for 100 complete oscillations. On my computer this time was $t = 293\ sec$ ($4'53''$), so the period of a single oscillation was:

$$T = \frac{293}{100} = 2.93\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

[4] I've also dropped the friction to zero and taken at the maximum value the rigidity of the spring that connects the free mass at the fixed point. In the pendulum theory the connection between the mass and the fixed point should be perfectly rigid.

$$T = 2.93\,sec \cdot 110.81 \frac{frm}{sec} = 324.67\,frm$$

Applying (3.6) I finally found:

$$g(20) = 0.140067762\ \frac{pxl}{frm^2}$$

For now we will ignore the question of significant digits. We will discuss this at the end of the chapter.

- ***Experiment 2***

Using Experiment 2 it was possible to find the value of the gravitational acceleration constant g (in pxl/frm^2) when the cursor position is $p_g = 29$. The length of the pendulum in this case is $l = 374\ pxl$.

I measured the time it took for 100 complete oscillations. On my computer this time was $t = 202\ sec$ ($3'22''$), so the period of a single oscillation was:

$$T = \frac{202}{100} = 2.02\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 2.02\,sec \cdot 110.81 \frac{frm}{sec} = 223.84\,frm$$

Applying (3.6) I finally found:

$$g(29) = 0.294693591\ \frac{pxl}{frm^2}$$

- ***Experiment 3***

Using Experiment 3 it was possible to find the value of the gravitational acceleration constant g (in pxl/frm^2) when the cursor position is $p_g = 40$. The length of the pendulum in this case is $l = 375\ pxl$.

I measured the time it took for 150 complete oscillations. On my computer this time was $t = 220\ sec$ ($3'40''$), so the period of a single oscillation was:

$$T = \frac{220}{150} = 1.47\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 1.47\ sec \cdot 110.81 \frac{frm}{sec} = 162.52\ frm$$

Applying (3.6) I finally found:

$$g(40) = 0.560493075\ \frac{pxl}{frm^2}$$

• ***Experiment 4***

Using Experiment 4 it was possible to find the value of the gravitational acceleration constant g (in pxl/frm^2) when the cursor position is $p_g = 50$. The length of the pendulum in this case is $l = 375\ pxl$.

I measured the time it took for 150 complete oscillations. On my computer this time was $t = 176\ sec$ ($2'56''$), so the period of a single oscillation was:

$$T = \frac{176}{150} = 1.17\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 1.17\ sec \cdot 110.81 \frac{frm}{sec} = 130.02\ frm$$

Applying (3.6) I finally found:

$$g(50) = 0.875770431\ \frac{pxl}{frm^2}$$

• ***Experiment 5***

Using Experiment 5 was possible to get the value of the gravitational acceleration constant g (in pxl/frm^2) when the cursor position is $p_g = 60$. The length of the pendulum in this case is $l = 376\ pxl$.

I measured the time it took for 200 complete oscillations. On my computer this time was $t = 196\ sec$ ($3'16''$), so the period of a single oscillation was:

$$T = \frac{196}{200} = 0.98 \ sec$$

By using the *time conversion ratio* (2.1) the period *T* becomes:

$$T = 0.98\,sec \cdot 110.81 \frac{frm}{sec} = 108.59 \ frm$$

Applying (3.6) I finally found:

$$g(60) = 1.25874431 \ \frac{pxl}{frm^2}$$

• Results Analysis

Displaying the above results in a graph in which the horizontal axis represents the cursor position p_g and the vertical axis represents the gravitational acceleration constant *g*, we get the curve of Figure 3.3:

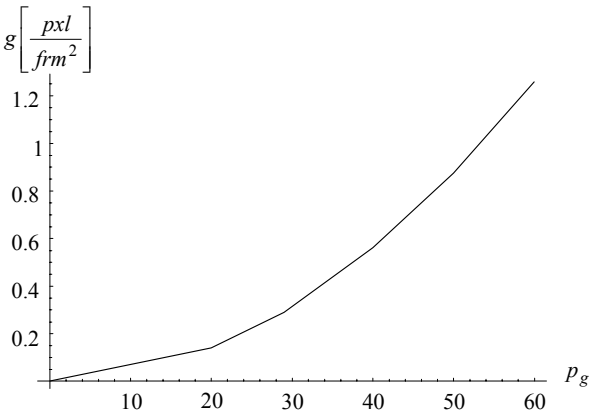

Figure 3.3: gravity trend

Looking the curve of Figure 3.3 we can immediately see that the relation between *g* and p_g is not linear. Since the curve looks more similar to a parabola, we can try to calculate the ratio between *g* and p_g^2. The following table shows the result of this calculation for all the above experiences:

Table 3.1: Ratio g/p_g^2 for the above experiences

p_g	$g\left[\frac{pxl}{frm^2}\right]$	$\frac{g}{p_g^2}$
20	0.140067762	0.000350169
29	0.294693591	0.000350409
40	0.560493075	0.000350308
50	0.875770431	0.000350308
60	1.25874431	0.000349651

Since the ratio g/p_g^2 is practically constant for all 5 of the above virtual experiences, we can affirm that between g and p_g there is a quadratic proportionality:

$$g(p_g) = g_P \cdot p_g^2 \qquad \left[\frac{pxl}{frm^2}\right] \tag{3.8}$$

In the formula above I've introduced the *gravity parameter* g_p which is a constant parameter implemented in the sodaconstructor applet. If I were extremely precise I would find the gravity parameter g_p by means of a quadratic interpolation of the above data, but remembering that all of this is just play, I will restrict myself to the calculation of the medium value of g_p. From Table 3.1 we can find the following medium value for g_p:

$$g_p = 0.000350169 \qquad \left[\frac{pxl}{frm^2 \cdot p_g^2}\right] \tag{3.9}$$

Figure 3.4 displays the numerical data of Table 3.1 and the continuous curve made from (3.8) and (3.9). As can clearly be seen, in the range of p_g that was used in the experiments ($p_g \in [20, 60]$) there is a perfect overlapping.

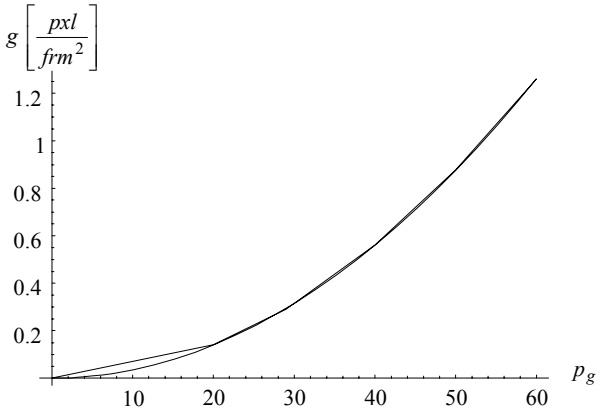

Figure 3.4: Overlapping between numerical data and theoretical curve

3.2 *THE STIFFNESS (STATIC METHOD)*

The main components of the sodaconstructor are springs and muscles. These components have elastic properties that are defined by a physical quantity called *stiffness*. To understand what the elastic properties of springs exactly are and how these properties are represented by stiffness, we could make some observations of springs in the real world. We all know what real springs are and what characteristics they possess. We know, for example, that a spring changes length only when force is applied to its ends, and that when this force stops the spring returns to its original length. We also know that the force necessary to pull a spring increases with the spring's extension. Finally, we know that two springs of different strengths subjected to the same forces have different extensions.

These observations show the basic concepts of elasticity and stiffness. We will say that a body is an elastic body if its deformations vanish when their causes are removed. We will say that a body is a linear elastic body if its deformations are directly proportional to the forces applied to the body. Finally we will define stiffness as how well an elastic body can maintain its original shape (or length if we are specifically speaking of a spring) when it is subjected to external forces.

For springs, all these characteristics can be mathematically defined in a very simple way.

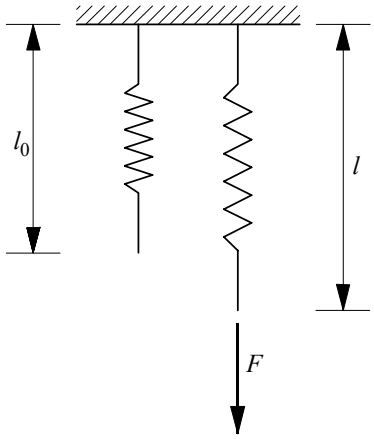

Figure 3.5: Extension of a spring subjected to traction by a force *F*

Figure 3.5 shows a spring before and after the action of a traction force *F*. Calling l_0 the length of the spring at rest and *l* the length of the spring after extended, we will define the quantity of the spring's extension:

$$\Delta l = l - l_0 \tag{3.10}$$

By definition, the extension of a spring will be positive if its final length *l* is greater than the initial length l_0. The extension of a spring will be negative if its final length *l* is less than the initial length l_0.

If the spring is a linear elastic spring (like the springs in the sodaconstructor applet), then the spring's extension Δl will be directly proportional to the force's intensity *F* by means of the relation:

$$F = k \cdot \Delta l \tag{3.11}$$

This law is known as Hooke's law[5]. The constant *k* that appears in (3.11) is the stiffness of the spring and is a physical quantity that is always positive. Its value is representative of the spring's strength. Using (3.11), we can say:

[5] Robert Hooke was an English scientist of seventeenth century, contemporaneous of Isaac Newton. It seems that the two scientists weren't particularly fond of each other. It is ironic that in our beloved sodaconstructor applet their laws live together harmoniously.

$$\Delta l = \frac{F}{k} \tag{3.12}$$

It is easy to understand that under the same force, springs with high values of stiffness *k* will have extensions smaller than those of springs with low stiffness.

From (3.11) we can also see that if the force is positive, its effect on the spring will be positive (extension), while if the force is negative, its effect on the spring will be negative (compression).

The stiffness *k* that appears in (3.11) and (3.12) is the same physical quantity that can be changed with the cursor *k* (see Fig. 3.1) in the sodaconstructor applet. There is one more thing I must emphasize about the sodaconstructor's virtual springs: all springs, regardless of their initial length, have the same stiffness[6]. In the current version of the applet it is impossible to have springs of independent stiffness in the same simulation.

By means of static virtual experiments, the expression (3.11) will allow us to determine the value of the stiffness *k* depending on the position of the corresponding cursor. Obviously, in order to use (3.11) to find the stiffness *k*, we will need a static force *F*. But how can we create a static force? We can simply use the force of gravity. We know that a mass *m* in a constant gravitational field has a weight given by the expression (3.1). Therefore, in order to create a constant force in our experiments, we will use the virtual weight of free masses in the sodaconstructor applet.

It might be important at this point to say something about the sodaconstructor's free masses. By means of very simple virtual experiments it is possible to prove that all free masses are equal[7], so we will have just one value of mass *m* for all free masses. At the moment, we will leave this value unknown. In the Chapter 8 we will see why it is possible to avoid having to assume any specific value for *m*.

[6] i.e. all springs, regardless of their initial length, will be subject to the same extension under the same force.

[7] I will leave to you the pleasure of devising some simple experiments.

• ***The Static Experiments***

Just as we did with the gravity *g*, our objective will be to find how the spring's stiffness *k* changes according to the position of the corresponding cursor; i.e. our objective will be to discover the law:

$$k = k(p_k) \tag{3.13}$$

In order to find this law we will apply the weight of a free mass to springs of different lengths[8], using different fixed values of stiffness. Therefore, the constant force *F* in the relation (3.11) for us will be the weight *W* of a free mass given by (3.1). Then we will have:

$$F = W = m \cdot g = k \cdot \Delta l \tag{3.14}$$

so that:

$$k = \frac{g}{\Delta l} \cdot m \tag{3.15}$$

In (3.15), we know the value of *g* thanks to (3.8) and (3.9), and we will find the extension Δl by measuring it. As I've said before, at the moment we will leave the value of *m* unknown, so it will be helpful to define the following physical quantity:

$$\bar{k} = \frac{k}{m} = \frac{g}{\Delta l} \tag{3.16}$$

which is independent of the value of *m*. We will call $\bar{k}$ *specific stiffness.*
Thanks to the introduction of specific stiffness we can write (3.15) as:

$$k = \bar{k} \cdot m \tag{3.17}$$

[8] I've used springs of different lengths to prove without any doubt that the stiffness of a spring is absolutely independent of its initial length.

• ***Experiment 6***

In Experiment 6 the extensions of springs of different lengths have been measured with the stiffness cursor position equal to $p_k = 20$. The force F has been varied by changing the position of the gravity cursor from $p_g = 30$ to $p_g = 100$ by increments of 10. For each setting of gravity, the extension of the springs has been measured and the results reported in the following Table 3.2:

Table 3.2: Spring's extension for different values of gravity when $p_k = 20$

p_g	$g\left[\frac{pxl}{frm^2}\right]$	$\Delta l\ [pxl]$	$\frac{g}{\Delta l}\left[\frac{1}{frm^2}\right]$
30	0.3151521	18	0.01750845
40	0.5602704	32	0.01750845
50	0.8754225	50	0.01750845
60	1.2606084	72	0.01750845
70	1.7158281	98	0.01750845
80	2.2410816	128	0.01750845
90	2.8363689	162	0.01750845
100	3.50169	200	0.01750845

Using the above results, we find the average value of the specific stiffness for the cursor position $p_k = 20$:

$$\bar{k}(20) = 0.01750845\ \frac{1}{frm^2} \tag{3.18}$$

• ***Experiment 7***

In Experiment 7 the extensions of springs of different lengths have been measured with the stiffness cursor position equal to $p_k = 25$. The force F has been varied by changing the

position of the gravity cursor from $p_g = 30$ to $p_g = 100$ by increments of 10. For each setting of gravity, the extension of the springs has been measured and the results reported in the following Table 3.3:

Table 3.3: Spring's extension for different values of gravity when $p_k = 25$

p_g	$g\left[\frac{pxl}{frm^2}\right]$	$\Delta l\ [pxl]$	$\frac{g}{\Delta l}\left[\frac{1}{frm^2}\right]$
30	0.3151521	11	0.028650191
40	0.5602704	20	0.028013520
50	0.8754225	32	0.027356953
60	1.2606084	46	0.027404530
70	1.7158281	62	0.027674647
80	2.2410816	81	0.027667674
90	2.8363689	103	0.027537562
100	3.50169	128	0.027356953

Using the above results, we find the average value of the specific stiffness for the cursor position $p_k = 25$:

$$\bar{k}(25) = 0.027707754\ \frac{1}{frm^2} \tag{3.19}$$

• ***Experiment 8***

In Experiment 8 the extensions of springs of different lengths have been measured with the stiffness cursor position equal to $p_k = 30$. The force *F* has been varied by changing the position of the gravity cursor from $p_g = 30$ to $p_g = 100$ by increments of 10. For each setting of gravity, the extension of the springs has been measured and the results reported in the following Table 3.4:

Table 3.4: Spring's extension for different values of gravity when p_k = 30

p_g	$g\left[\frac{pxl}{frm^2}\right]$	$\Delta l\ [pxl]$	$\frac{g}{\Delta l}\left[\frac{1}{frm^2}\right]$
30	0.3151521	8	0.039394013
40	0.5602704	14	0.040019314
50	0.8754225	22	0.039791932
60	1.2606084	32	0.039394013
70	1.7158281	43	0.039902979
80	2.2410816	56	0.040019314
90	2.8363689	72	0.039394013
100	3.50169	88	0.039791932

Using the above results, we find the average value of the specific stiffness for the cursor position $p_k = 30$:

$$\bar{k}(30) = 0.039713439\ \frac{1}{frm^2} \tag{3.20}$$

• ***Results Analysis***

In this case we have the values of the specific stiffness for only three positions on the cursor p_k, so it will be impossible to get as good a curve as we did with the gravity experiments. Therefore we will limit ourselves to a simple analysis of the above numerical data. Based on the results of our gravity experiments, it is reasonable to guess that the relationship between the specific stiffness and p_k is once again quadratic. In order to verify this hypothesis we will calculate the ratio $\bar{k}(p_k)/p_k^2$ for each of the three results reported in (3.18), (3.19) and (3.20).

Table 3.5: Ratio g/p_k^2 for the above experiments

p_k	$\bar{k}\left[\frac{1}{frm^2}\right]$	$\frac{\bar{k}}{p_k^2}\left[\frac{1}{frm^2}\right]$
20	0.01750845	0.000043771
25	0.027707754	0.000044332
30	0.039713439	0.000044126

Because the results of the ratio $\bar{k}(p_k)/p_k^2$ are approximately constant, we can assume with very little doubt the following rule for specific stiffness:

$$\bar{k}(p_k) = \bar{k}_p \cdot p_k^2 \tag{3.21}$$

in which has been introduced the parameter $\bar{k}_p$ that we will call the *stiffness parameter*. Just as with the gravity parameter g_p, the stiffness parameter $\bar{k}_p$ is a constant parameter implemented in the sodaconstructor applet. Its medium value is (see Table 3.5):

$$\bar{k}_p = 0.000044077\left[\frac{1}{frm^2}\right] \tag{3.22}$$

Therefore the rule for the stiffness *k*, thanks to (3.17) and (3.21), will be:

$$k(p_k) = \bar{k}(p_k)\cdot m = \bar{k}_p \cdot m \cdot p_k^2 \tag{3.23}$$

3.3 *THE STIFFNESS (DYNAMIC METHOD)*

There is another and more accurate method of determining the stiffness of a spring. This method is very similar to the dynamic method that we used for the gravity experiments. By simply measuring the period of the oscillation of a spring connected to one mass, it is possible to calculate the spring's stiffness. In the following section I will restrict myself to explain just

the main relation that we will use. A more detailed explanation of this method can be found in Appendix B.

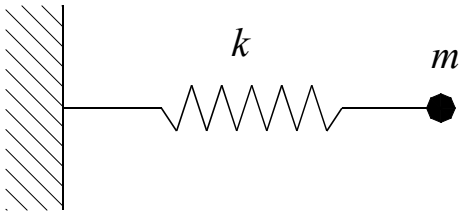

Figure 3.6: System spring-mass

Calling *k*, *m* and *T* respectively the stiffness of a spring, the value of the mass connected with the spring and the period of a complete oscillation of the system spring-mass (see Fig. 3.6), it is possible to show that the following relation is valid:

$$k = \frac{4\pi^2}{T^2} m \tag{3.24}$$

Using this relation, it is possible to get the stiffness of a spring by measuring the period *T*. Taking into account (3.17), from (3.24) follows immediately the expression for the specific stiffness $\bar{k}$:

$$\bar{k} = \frac{4\pi^2}{T^2} \tag{3.25}$$

Obviously, like in the gravity determination, with friction equal to zero the oscillations of the spring will last forever. Therefore, to get a more accurate reading, we will measure the duration of more than one oscillation.

• ***Experiment 9***

In Experiment 9, it was possible to find the value of the specific stiffness $\bar{k}$ (in $1/frm^2$) when the cursor position was $p_k = 6$. The duration of 100 complete oscillations has been

measured. In my computer this time was $t = 142\ sec$ $(2'22'')$, so the period of a single oscillation was:

$$T = \frac{142}{100} = 1.42\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 1.42\ sec \cdot 110.81 \frac{frm}{sec} = 157.35\ frm$$

Applying (3.25) I finally found:

$$\bar{k}(6) = 0.001594502\ \frac{1}{frm^2}$$

• ***Experiment 10***

In Experiment 10, it was possible to find the value of the specific stiffness $\bar{k}$ (in $1/frm^2$) when the cursor position was $p_k = 8$. The duration of 150 complete oscillations has been measured. In my computer this time was $t = 160\ sec$ $(2'40'')$, so the period of a single oscillation was:

$$T = \frac{160}{150} = 1.07\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 1.07\ sec \cdot 110.81 \frac{frm}{sec} = 118.20\ frm$$

Applying (3.25) I finally found:

$$\bar{k}(8) = 0.002825819\ \frac{1}{frm^2}$$

• ***Experiment 11***

In Experiment 11, it was possible to find the value of the specific stiffness $\bar{k}$ (in $1/frm^2$) when the cursor position was $p_k = 10$. The duration of 150 complete oscillations has been measured. In my computer this time was $t = 128\ sec$ $(2'08'')$, so the period of a single oscillation was:

$$T = \frac{128}{150} = 0.85\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 0.85\,sec \cdot 110.81 \frac{frm}{sec} = 94.56\ frm$$

Applying (3.25) I finally found:

$$\bar{k}(10) = 0.004415343\ \frac{1}{frm^2}$$

• ***Experiment 12***

In Experiment 12, it was possible to find the value of the specific stiffness $\bar{k}$ (in $1/frm^2$) when the cursor position was $p_k = 12$. The duration of 150 complete oscillations has been measured. In my computer this time was $t = 107\ sec$ $(1'47'')$, so the period of a single oscillation was:

$$T = \frac{107}{150} = 0.71\ sec$$

By using the *time conversion ratio* (2.1) the period T becomes:

$$T = 0.71\,sec \cdot 110.81 \frac{frm}{sec} = 79.04\ frm$$

Applying (3.25) I finally found:

$$\bar{k}(12) = 0.006318541\ \frac{1}{frm^2}$$

• ***Experiment 13***

In Experiment 13, it was possible to find the value of the specific stiffness $\bar{k}$ (in $1/frm^2$) when the cursor position was $p_k = 14$. The duration of 150 complete oscillations has been measured. In my computer this time was $t = 91\ sec$ $(1'31'')$ so the period of a single oscillation was:

$$T = \frac{91}{150} = 0.61\ sec$$

By using the *time conversion ratio* (2.1) the period *T* becomes:

$$T = 0.61\,sec \cdot 110.81 \frac{frm}{sec} = 67.22\,frm$$

Applying (3.25) I finally found:

$$\bar{k}(14) = 0.008735777\ \frac{1}{frm^2}$$

• ***Results Analysis***

Displaying the above results in a diagram in which the horizontal axis represents the cursor position p_k and the vertical axis represents the specific stiffness $\bar{k}$, we get the curve in Figure 3.7:

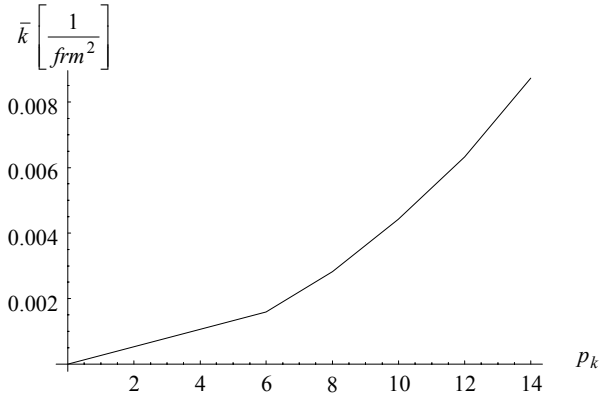

Figure 3.7: specific stiffness trend

First of all, it is possible to see how the curve in Figure 3.7 confirms that there isn't a direct proportionality between $\bar{k}$ and p_k. Therefore, also taking into account the result obtained in the previous section 3.2, we will try to calculate the ratio between $\bar{k}$ and p_k^2 for the above results. If there were no errors in the experiments or calculations, then the value for the ratio $\bar{k}/p_k^2$ can't be much different than the value obtained in (3.22). The results of this ratio for all the above experiments is shown in the following table:

Table 3.6: Ratio $\bar{k}/p_k^2$ for the above experiments

p_k	$\bar{k}\left[\frac{1}{frm^2}\right]$	$\frac{\bar{k}}{p_k^2}\left[\frac{1}{frm^2}\right]$
6	0.001594502	0.000044292
8	0.002825819	0.000044153
10	0.004415343	0.000044153
12	0.006318541	0.000043879
14	0.008735777	0.000044570

Since the ratio $\bar{k}/p_k^2$ is practically constant for all five of the above virtual experiments, we have further evidence of the validity of the experimental relation (3.21). Moreover, it is possible to see that all the values of the ratio $\bar{k}/p_k^2$ are practically the same of the value (3.22) obtained by means of the static experiments. Since the dynamic experiment gives more accurate results, for the stiffness parameter we will take the following medium value:

$$\bar{k}_p = 0.000044209\left[\frac{1}{frm^2}\right] \tag{3.26}$$

The quasi-perfect coincidence between the values of $\bar{k}_p$ obtained by means of two different methods of experiments is proof of the validity of this section of virtual experiments.

3.4 *THE FRICTION*

The friction discussed here is the friction that a body meets while in movement in a fluid. Therefore, this kind of friction is related to the viscosity of the medium in which the body is moving. This friction has essentially two characteristics: the first is that the force of the friction is always proportional to the body's velocity; the second is that the force of the friction is such that its effect is always in opposition of the movement. In order to mathematically define this kind of friction we will introduce a new physical quantity that in

the following parts of this paper will be called the *damping constant*[9] and that will be indicated with the letter *f*. It is this quantity that we frequently change in our models by adjusting the second cursor of Figure 3.1.

Since velocity isn't a scalar quantity, but rather a vectorial quantity, when defining velocity it is insufficient to specify just its intensity (for example expressed in m/sec). It is also necessary to specify its direction. Therefore, velocity, like all other vectorial quantities, can be represented by vectors[10] (see Figure 3.8 *a*). Force, like velocity and acceleration, is a vectorial quantity, so it can also be represented by vectors. Here we are referring to forces from a body's motion in a viscous fluid. So, taking into account what was said earlier about these forces, we can represent them by means of a vector which has: intensity equal to the velocity's intensity multiplied by the damping constant *f*, and the opposite direction as the velocity (see Figure 3.8 *b*).

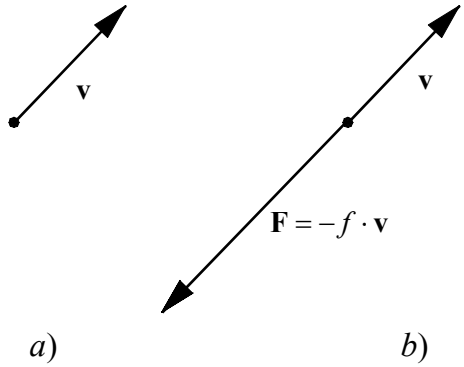

Figure 3.8: Relation between velocity and friction force.

Obviously, the force of friction will tend to reduce the motion of a body, so that if anything causes motion (for example something like a muscle), the body eventually will stop.

The main way to determine the damping constant *f* is to study the free oscillation of a spring-mass system (like in the dynamic determination of the stiffness) in an environment of viscous

[9] In sodalanguage, this parameter is usually simply called *friction*. Nevertheless, here, in order to avoid any confusion between the different kinds of frictions that are encountered in the real world, I have named this parameter more appropriately the damping constant.

[10] It is possible to represent a vector by means of an arrow. The arrow's length will represent in the appropriate scale the vector's intensity; the orientation of the arrow will define the vector's direction.

friction: the damped free vibration. A detailed explanation of this method can be found in Appendix B, so for now, we will just discuss the main relations that will be used.

Leaving a spring-mass system like the one represented in Figure 3.6 free to oscillate in presence of viscous friction, and calling $x(t)$ the mass displacement in respect to the quiet position at the generic instant *t*, it is possible to report in a Cartesian diagram the evolution of the system. By reporting the mass displacement *x* in the vertical axis and the time *t* in the horizontal axis, as we will see in Appendix B, will yield a graphic similar to the following Figure 3.9.

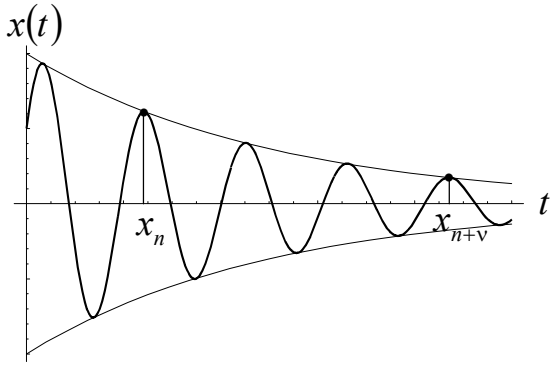

Figure 3.9: Effect of viscous damping on a free vibration.

Considering two positive peaks between a number of ν complete cycles of oscillations and calling x_n and $x_{n+\nu}$ their respective values it is possible to calculate the following logarithmic parameter[11]:

$$\delta = \ln\left(\frac{x_n}{x_{n+\nu}}\right) \tag{3.27}$$

Using this, it is possible to get the damping constant *f* value:

[11] We are considering the natural logarithm; i.e. the inverse function of the exponential function e^x.

$$f = \frac{2\sqrt{k\,m}}{\sqrt{1+\left(\frac{2\nu\pi}{\delta}\right)^2}} \tag{3.28}$$

In (3.28) we have respectively indicated by k and m the spring's stiffness and the mass value. Therefore, taking into account the position (3.17) about the specific stiffness of a spring, we can rewrite the relation (3.28) as:

$$f = \frac{2\sqrt{\bar{k}}}{\sqrt{1+\left(\frac{2\nu\pi}{\delta}\right)^2}}\,m \tag{3.29}$$

Just as with stiffness, it is possible here to introduce a new parameter that we will call *specific damping constant* $\bar{f}$ which doesn't need the free mass value m:

$$\begin{cases} f = \bar{f}\cdot m \\ \bar{f} = \dfrac{2\sqrt{\bar{k}}}{\sqrt{1+\left(\frac{2\nu\pi}{\delta}\right)^2}} \end{cases} \tag{3.30}$$

Thanks to the second formula of (3.30) we will be able to find the sodaconstructor's law of variation with the friction cursor's position p_f of the specific damping $\bar{f}$.

• ***Experiment 14***

In Experiment 14, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 1$ and $p_k = 6$ respectively. The value of the specific damping $\bar{f}$ should be independent of the stiffness, but in order to verify this independence, we will repeat every experiment (i.e. for a fixed value of the cursor position p_f) with two different values of the specific stiffness.

The peak values of the first and 73rd cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 73 \\ \nu = 72 \\ x_n = x_1 = 291\ pxl \\ x_{n+\nu} = x_{73} = 177\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{291}{177}\right) = 0.497173535$$

Moreover, using the relations (3.21) and (3.26) in order to get the value of the specific stiffness, it follows:

$$\bar{k}(6) = 0.000044209 \cdot 6^2 = 0.001591524 \frac{1}{frm^2}$$

Finally, taking into account the previous values, from the 2nd formula (3.30) it follows:

$$\bar{f}(1) = \frac{2\sqrt{0.001591524}}{\sqrt{1 + \left(\frac{2 \cdot 72 \cdot \pi}{0.497173535}\right)^2}} = 0.000087686 \frac{1}{frm}$$

• ***Experiment 15***

In Experiment 15, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 1$ and $p_k = 8$ respectively.

The peak values of the first and 101st cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 101 \\ \nu = 100 \\ x_n = x_1 = 290\ pxl \\ x_{n+\nu} = x_{101} = 173\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{290}{173}\right) = 0.516589328$$

For the specific stiffness we have the value:

$$\bar{k}(8) = 0.000044209 \cdot 8^2 = 0.002829376 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(1) = \frac{2\sqrt{0.002829376}}{\sqrt{1+\left(\frac{2 \cdot 100 \cdot \pi}{0.516589328}\right)^2}} = 0.000087466 \frac{1}{frm}$$

The results of the last two experiments establish that, neglecting the inevitable small errors, the specific damping is independent of the specific stiffness.

The medium value for the specific damping when $p_f = 1$ will be:

$$\bar{f}(1) = 0.000087576 \frac{1}{frm}$$

• ***Experiment 16***

In Experiment 16, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 2$ and $p_k = 6$ respectively.

The peak values of the first and 25th cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 25 \\ \nu = 24 \\ x_n = x_1 = 311\ pxl \\ x_{n+\nu} = x_{25} = 160\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{311}{160}\right) = 0.664619097$$

For the specific stiffness we have the value:

$$\bar{k}(6) = 0.001591524 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(2) = \frac{2\sqrt{0.001591524}}{\sqrt{1 + \left(\frac{2 \cdot 24 \cdot \pi}{0.664619097}\right)^2}} = 0.000351653 \frac{1}{frm}$$

• ***Experiment 17***

In Experiment 17, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 2$ and $p_k = 8$ respectively.

The peak values of the first and 33rd cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 33 \\ \nu = 32 \\ x_n = x_1 = 312\ pxl \\ x_{n+\nu} = x_{33} = 161\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{312}{161}\right) = 0.661598823$$

For the specific stiffness we have the value:

$$\bar{k}(8) = 0.002829376 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(2) = \frac{2\sqrt{0.002829376}}{\sqrt{1 + \left(\frac{2 \cdot 32 \cdot \pi}{0.661598823}\right)^2}} = 0.000350056 \frac{1}{frm}$$

The medium value for the specific damping when $p_f = 2$ will be:

$$\bar{f}(2) = 0.000350855 \frac{1}{frm}$$

• ***Experiment 18***

In Experiment 18, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 3$ and $p_k = 6$ respectively.

The peak values of the first and 21st cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 21 \\ \nu = 20 \\ x_n = x_1 = 306\ pxl \\ x_{n+\nu} = x_{21} = 88\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{306}{88}\right) = 1.246248287$$

For the specific stiffness we have the value:

$$\bar{k}(6) = 0.001591524 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(3) = \frac{2\sqrt{0.001591524}}{\sqrt{1 + \left(\frac{2 \cdot 20 \cdot \pi}{1.246248287}\right)^2}} = 0.000791243 \frac{1}{frm}$$

• ***Experiment 19***

In Experiment 19, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 3$ and $p_k = 8$ respectively.

The peak values of the first and 28th cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 28 \\ \nu = 27 \\ x_n = x_1 = 308\ pxl \\ x_{n+\nu} = x_{28} = 87\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{308}{87}\right) = 1.264191664$$

For the specific stiffness we have the value:

$$\bar{k}(8) = 0.002829376 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(3) = \frac{2\sqrt{0.002829376}}{\sqrt{1 + \left(\frac{2 \cdot 27 \cdot \pi}{1.264191664}\right)^2}} = 0.000792743 \frac{1}{frm}$$

The medium value for the specific damping when $p_f = 3$ will be:

$$\bar{f}(3) = 0.000791993 \frac{1}{frm}$$

• ***Experiment 20***

In Experiment 20, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 4$ and $p_k = 6$ respectively.

The peak values of the first and 18th cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 18 \\ \nu = 17 \\ x_n = x_1 = 298\ pxl \\ x_{n+\nu} = x_{18} = 46\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{298}{46}\right) = 1.86845209$$

For the specific stiffness we have the value:

$$\bar{k}(6) = 0.001591524\,\frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(4) = \frac{2\sqrt{0.001591524}}{\sqrt{1+\left(\frac{2\cdot 17\cdot \pi}{1.86845209}\right)^2}} = 0.001395479\,\frac{1}{frm}$$

• ***Experiment 21***

In Experiment 21, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 4$ and $p_k = 8$ respectively.

The peak values of the first and 23rd cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 23 \\ \nu = 22 \\ x_n = x_1 = 303\ pxl \\ x_{n+\nu} = x_{23} = 49\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{303}{49}\right) = 1.821912507$$

For the specific stiffness we have the value:

$$\bar{k}(8) = 0.002829376 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(4) = \frac{2\sqrt{0.002829376}}{\sqrt{1+\left(\frac{2 \cdot 22 \cdot \pi}{1.821912507}\right)^2}} = 0.001402047 \frac{1}{frm}$$

The medium value for the specific damping when $p_f = 4$ will be:

$$\bar{f}(4) = 0.001398763 \frac{1}{frm}$$

• ***Experiment 22***

In Experiment 22, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f = 5$ and $p_k = 6$ respectively.

The peak values of the first and 11th cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n = 1 \\ n + \nu = 11 \\ \nu = 10 \\ x_n = x_1 = 289\ pxl \\ x_{n+\nu} = x_{11} = 51\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta = \ln\left(\frac{289}{51}\right) = 1.734601055$$

For the specific stiffness we have the value:

$$\bar{k}(6) = 0.001591524 \frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(5)=\frac{2\sqrt{0.001591524}}{\sqrt{1+\left(\frac{2\cdot 10\cdot \pi}{1.734601055}\right)^2}}=0.002201866\frac{1}{frm}$$

• ***Experiment 23***

In Experiment 23, it was possible to find the value of the specific damping $\bar{f}$ (expressed in $1/frm$) when the cursor positions for friction and stiffness were $p_f=5$ and $p_k=8$ respectively.

The peak values of the first and 14th cycles of oscillations were measured, and the following values were obtained:

$$\begin{cases} n=1 \\ n+\nu=14 \\ \nu=13 \\ x_n=x_1=295\ pxl \\ x_{n+\nu}=x_{14}=55\ pxl \end{cases}$$

From (3.27) it follows:

$$\delta=\ln\left(\frac{295}{55}\right)=1.679642171$$

For the specific stiffness we have the value:

$$\bar{k}(8)=0.002829376\frac{1}{frm^2}$$

Finally, taking in account the previous values, from (3.30) it follows:

$$\bar{f}(5)=\frac{2\sqrt{0.002829376}}{\sqrt{1+\left(\frac{2\cdot 13\cdot \pi}{1.679642171}\right)^2}}=0.002187142\frac{1}{frm}$$

The medium value for the specific damping when $p_f=5$ will be:

$$\bar{f}(5)=0.002194504\frac{1}{frm}$$

• ***Results Analysis***

As with the gravity and stiffness experiments, looking at the following Table 3.7, which contains the values obtained in the previous virtual experiments, it is possible to establish that the specific damping $\bar{f}$ is related to the cursor position p_f by means of a quadratic proportionality.

Table 3.7: ratio between the specific damping and the cursor position

p_f	$\bar{f}\left[\frac{1}{frm}\right]$	$\frac{\bar{f}}{p_f^2}\left[\frac{1}{frm}\right]$
1	0.000087576	0.000087576
2	0.000350855	0.000087714
3	0.000791993	0.000087999
4	0.001398763	0.000087423
5	0.002194504	0.000087780

Just as with the previous physical quantities, it is possible to assume for the specific damping the following law:

$$\bar{f}\left(p_f\right) = \bar{f}_p \cdot p_f^2 \quad \left[\frac{1}{frm}\right] \tag{3.31}$$

in which the value of the *damping parameter* $\bar{f}_p$ can be obtained as medium value from the values of the ratio $\bar{f} \big/ p_f^2$ reported in the above Table 3.7. We will assume for the damping parameter the following value:

$$\bar{f}_p = 0.000087698 \ \frac{1}{frm} \tag{3.32}$$

In the following graph are reported both the experimental values of the specific damping and the corresponding interpolation value obtained from (3.31) and (3.32).

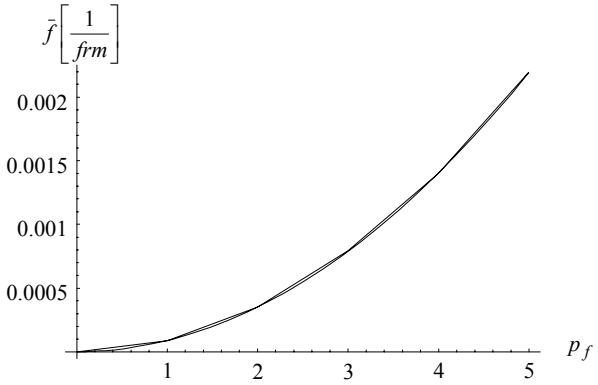

Figure 3.10: Experimental and interpolating value for the specific damping.

3.5 *COLLISIONS*

The only collisions that exist in the sodaconstructor applet are collisions between masses and walls[12]. There aren't collisions between masses themselves: these entities, even though they appear to have a particular size, are mathematically just points (i.e. couples of Cartesian coordinates). For this reason we will avoid speaking in depth about this phenomenon. The only thing that we need to understand is the way in which collisions between masses and rigid walls occur.

There are two classic kinds of collisions: *elastic* and *non-elastic*. In elastic collisions, the intensity of the velocity of a mass immediately before and after its impact on a rigid wall is exactly the same. Only the direction of the velocity changes, depending on the angle of impact. In non-elastic collisions, the velocity of the mass immediately after the impact drops to zero. It is as if the wall were sprinkled with glue: the mass remains attached to the wall.

In reality, neither of these types of impacts exists. If elastic collisions existed in the real world, then, in principle, it would be possible to have a ball that bounces on the ground forever. Moreover, in reality, non-elastic collisions always result in at least a small recoil of the mass.

[12] By *wall* I mean the walls, the ground and the ceiling.

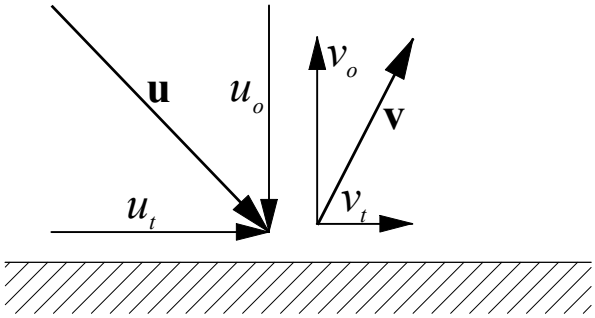

Figure 3.11: velocities before and after an impact of a mass on a rigid wall.

We will define **u** and **v**, respectively, as the velocities of a mass immediately before and after its collision with a rigid wall. In reference to the wall's surface, we will indicate with u_o and v_o their orthogonal components, and with u_t and v_t their tangential components (see Figure 3.11). It is possible to see in Figure 3.11 that the orthogonal components of velocities have the opposite direction before and after the collision, while the tangential components of velocities have the same direction before and after the collision. What about the values of velocity?

Since, as mentioned earlier, collisions in the real world will never be perfectly elastic or non-elastic, in order to discuss mathematically exactly what happens during the impact of a mass on a rigid wall, we will introduce the *coefficients of elastic restitution* c_t and c_o. Thanks to these coefficients, it is possible to write:

$$\begin{cases} v_t = c_t \cdot u_t \\ v_o = c_o \cdot u_o \end{cases} \tag{3.33}$$

Taking into account that in a realistic collision $0 < c_t, c_o < 1$, we can immediately understand that the value of velocity after the impact is always less than the value of velocity before the impact. Obviously, we would have a true elastic impact if the coefficients c_t and c_o were equal to 1, and we would have a true non-elastic impact if the coefficients c_t and c_o were equal to 0.

The question now is: what happens in the sodaconstructor applet? It easy to verify that in collisions between masses and walls the masses always lose part of their kinetic energy[13], so in the next part of this section we will investigate the coefficients of elastic restitution of walls in the sodaconstructor applet.

• ***The Coefficient of Elastic Restitution in the Orthogonal Impact***

Thanks to the virtual experiment on orthogonal impact, it is possible to see that, in absence of friction[14], the maximum height of the free falling mass after each bounce is always less than the previous. Calling *u* and *v* respectively the velocity of a mass immediately before and after its impact on the ground, in accord with (3.33), the coefficient of elastic restitution for the orthogonal impact will be:

$$c_o = \frac{v}{u} \tag{3.34}$$

In order to find the velocity of the mass immediately before the impact on the ground we will use the well-known laws for uniformly accelerated motion. Using the height h_1 from which the mass begins falling, on the ground the velocity will be:

$$u = \sqrt{2gh_1} \tag{3.35}$$

What about the velocity *v*? We already know that the velocity *v* will be less than the velocity *u*, so, inevitably, the height h_2 that the mass will reach after the first rebound must be less than the high h_1. It is possible to calculate the velocity *v* by simply measuring the height h_2 reached by the mass after the first rebound:

$$v = \sqrt{2gh_2} \tag{3.36}$$

[13] i.e. the physical quantity related to the velocity of masses.
[14] The damping in the previous section.

Taking into account (3.34), (3.35) and (3.36) we finally have:

$$c_o = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}} = \sqrt{\frac{h_2}{h_1}} \tag{3.37}$$

Measuring the heights h_1 and h_2 in the previous virtual experiment we find $h_1 = 422\ \ pxl$ and $h_2 = 238\ \ pxl$. Applying the (3.37) we get:

$$c_o = \sqrt{\frac{238}{422}} = 0.750986713$$

What does this mean? It means that in orthogonal impacts with the walls, free masses always lose about 25% of their velocity!

• ***The Coefficient of Elastic Restitution in the Tangential Impact***

In order to find the value of the coefficient of elastic restitution in tangential impacts, a virtual experiment has been devised examining a particular tangential impact. In order to aid the explanation of the experiment I have also realized the schematic Figure 3.12.

In this experiment, a mass with an initial horizontal velocity is free to fall in absence of friction. The mass's motion, in its parabolic falling, can be defined in two components: the vertical motion and the horizontal motion. The vertical motion will be a uniformly accelerated motion, while the horizontal motion will simply be a uniform motion[15].

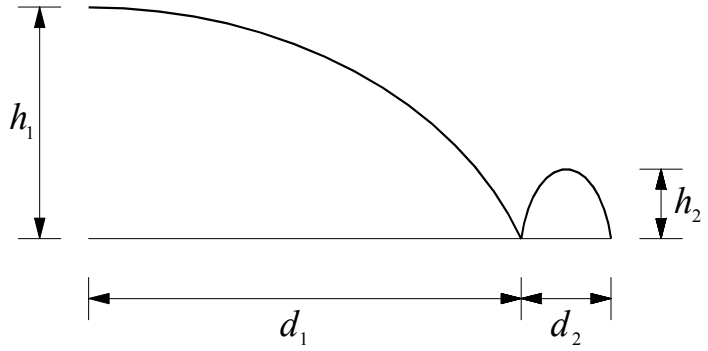

Figure 3.12: Scheme of the virtual experience

[15] i.e. the horizontal component of velocity is constant.

Calling h_1 the height from which the mass starts its parabolic falling, the time it will take for the mass to reach the ground is:

$$t_1 = \sqrt{\frac{2h_1}{g}} \tag{3.38}$$

Calling d_1 the horizontal displacement of the mass in this time, the horizontal component of the velocity will be:

$$u = \frac{d_1}{t_1} = d_1\sqrt{\frac{g}{2h_1}} \tag{3.39}$$

This is the tangential velocity of the mass before the impact on the ground.
In order to calculate the coefficient of elastic restitution in the tangential impact now we need the horizontal component of the velocity after the impact. It is possible to calculate this velocity by considering that after the impact the motion of the mass can again be defined in the two parts: the uniformly accelerated vertical motion and the uniform horizontal motion. Therefore, calling h_2 the maximum height reached by the mass after the first rebound, the time necessary to reach the ground again is:

$$t_2 = 2\sqrt{\frac{2h_2}{g}} \tag{3.40}$$

Calling d_2 the horizontal displacement of the mass during this time, the horizontal component of the velocity after the impact will be:

$$v = \frac{d_2}{t_2} = \frac{d_2}{2}\sqrt{\frac{g}{2h_2}} \tag{3.41}$$

In accord with the first formula of (3.33), and taking into account the velocities (3.39) and (3.41), the coefficient of elastic restitution for the tangential impact will be:

$$c_t = \frac{v}{u} = \frac{d_2}{2d_1}\sqrt{\frac{h_1}{h_2}} \tag{3.42}$$

From the virtual experiment mentioned earlier, it is possible to measure the following distances:

$$\begin{cases} d_1 = 377 \ \ pxl \\ d_2 = 57 \ \ pxl \\ h_1 = 422 \ \ pxl \\ h_2 = 238 \ \ pxl \end{cases}$$

From (3.42) follows:

$$c_t = \frac{57}{2 \cdot 377}\sqrt{\frac{422}{238}} = 0.100663322$$

What does this mean? This means that in tangential impacts with the walls, free masses always lose about 90% of their velocity! This fact is extremely enlightening (at least for me). Indeed, it is the low value of this coefficient that allows our models to walk. In the applet, there isn't friction on the ground in the real physical meaning of the word. If the value of this coefficient were greater, our models would walk with much difficulty. If this coefficient were equal to 1, our models would not be able to walk at all.

3.6 *FINAL RESULTS*

At the end of this long chapter I will simply summarize in a concise way the main results obtained by means of the previous virtual experiments.

We have defined with p_g, p_f and p_k the position of the cursors of Figure 3.1. These parameters can assume values from 0 to 107.

The law of variation of the acceleration constant *g* with the cursor's position p_g is:

$$g(p_g) = g_p \cdot p_g^2 \qquad \left[\frac{pxl}{frm^2}\right] \tag{3.43}$$

in which the value of the gravity parameter g_p experimentally obtained is:

$$g_p = 3.50169 \cdot 10^{-4} \qquad \left[\frac{pxl}{frm^2}\right] \tag{3.44}$$

The law of variation of the spring's stiffness *k* with the cursor's position p_k is:

$$\begin{cases} k(p_k) = \bar{k}(p_k) \cdot m \\ \bar{k}(p_k) = \bar{k}_p \cdot p_k^2 \qquad \left[\frac{1}{frm^2}\right] \end{cases} \tag{3.45}$$

in which the value of the stiffness parameter $\bar{k}_p$ experimentally obtained is:

$$\bar{k}_p = 4.4209 \cdot 10^{-5} \left[\frac{1}{frm^2}\right] \tag{3.46}$$

The law of variation of the damping *f* with the cursor's position p_f is:

$$\begin{cases} f(p_f) = \bar{f}(p_f) \cdot m \\ \bar{f}(p_f) = \bar{f}_p \cdot p_f^2 \qquad \left[\frac{1}{frm}\right] \end{cases} \tag{3.47}$$

in which the value of the damping parameter $\bar{f}_p$ experimentally obtained is:

$$\bar{f}_p = 8.7698 \cdot 10^{-5} \ \frac{1}{frm} \tag{3.48}$$

The components of the velocity before and after an impact between masses and walls are related by means of the equation:

$$\begin{cases} v_t = c_t \cdot u_t \\ v_o = c_o \cdot u_o \end{cases} \tag{3.49}$$

in which the values of the coefficients of elastic restitution experimentally obtained are:

$$\begin{cases} c_t \cong 0.10 \\ c_o \cong 0.75 \end{cases} \tag{3.50}$$

All of the above constants are certainly affected by inevitable errors in the measurements, so I would intuitively rely on just the first two significant digits. It would have been more thorough to calculate the errors using the well-known statistical techniques, but since this is all just play, I decided not to.

4. CINEMATIC

This chapter will discuss the main cinematic quantities[16] that we will encounter in the following chapter of this paper. These cinematic quantities are the *position vector*, the *velocity*, and the *acceleration*. All are related to the masses on the screen[17].

4.1 *SYSTEM OF COORDINATES*

The first thing we need is a system of coordinates. We will use a pair[18] of orthogonal Cartesian's axes with the origin fixed in the lower left corner of the sodaconstructor window. The horizontal axis will be the *x*-axis and its direction will be from the left to the right. The vertical axis will be the *y*-axis and its direction will be from the bottom to the top (see Figure 4.1). Thanks to this system of Cartesian's axes it will be possible to define the position of any free mass on the screen with a pair of Cartesian coordinates.

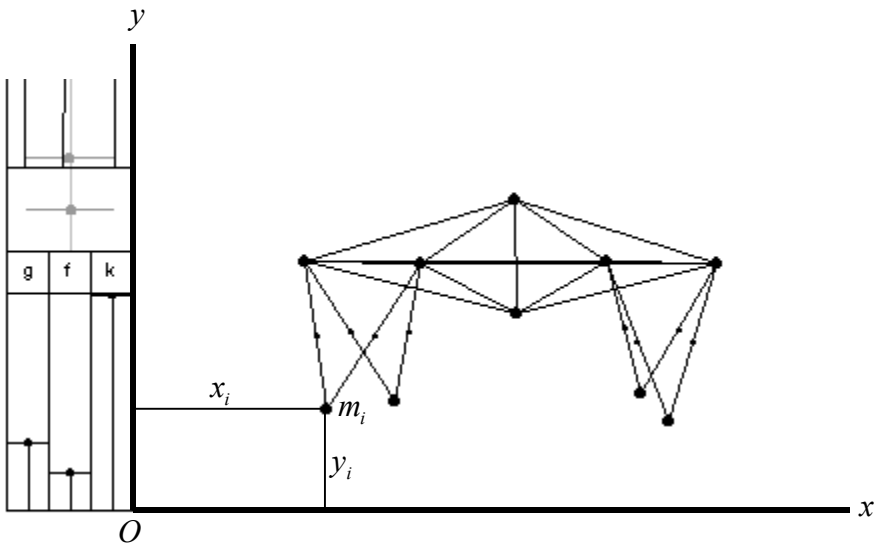

Figure 4.1: System of reference adopted in the rest of the paper.

[16] The quantities related to the motion of masses.
[17] From now on we will talk exclusively about elements of the sodaconstructor applet.
[18] Only two are needed because, as we know, the sodaconstructor is only a two-dimensional application of the main laws of mechanics.

4.2 *THE POSITION VECTOR*

If N is the integer number of masses on the screen, each mass will be characterized by an integer index included in the interval $[1,\ N]$. So we will have:

$$m_1,\ m_2,\ m_3,\ \cdots\ ,\ m_i,\ \cdots\ ,\ m_N$$

When we talk about a generic mass on the screen its index will be simply represented by the letter i. So, very often, we will talk about the generic mass m_i.

Each mass on the screen will be located by a pair of Cartesian coordinates. In order to avoid any confusion, each pair of coordinates will be characterized by the same index of the mass to which they refer. So the generic mass m_i will be located on the screen by the coordinates $x_i, y_i\ \ (i = 1, 2, \ldots, N)$ (see Fig. 4.1).

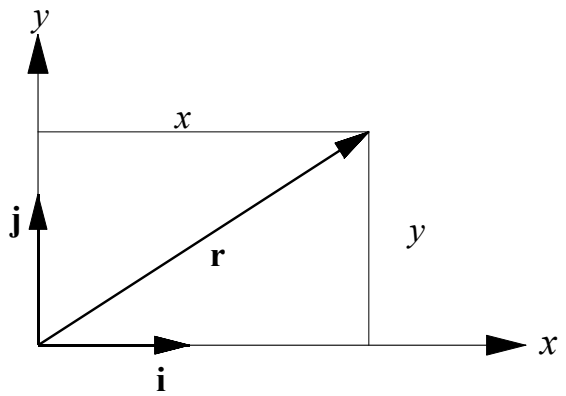

Figure 4.2: The position vector for the Cartesian point x, y.

In modern vector analysis a generic point in the space is located by a position vector. The Cartesian point (x, y) is given by the vector joining it to the origin of the coordinates (see Fig 4.2). This vector can be written as:

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \tag{4.1}$$

where **r** is the position vector of the Cartesian point (x, y), and **i** and **j** are the unit vectors[19] in the x and y directions. Therefore the position vector for the generic mass m_i will be represented by:

$$\mathbf{r}_i = x_i\,\mathbf{i} + y_i\,\mathbf{j} \tag{4.2}$$

4.3 *THE VELOCITY*

The examination of the movement of a mass requires two other cinematic quantities in addition to the position vector. These are the velocity and the acceleration. In order to define these quantities we will introduce the trajectory of a mass. This is the path obtained by joining in sequence all points in space crossed by the mass in its motion.

The trajectory could be represented by means of the vectorial function of time represented by the vector position $\mathbf{r}(t)$. This vectorial function of time locates the mass position at the generic instant of time *t*. Obviously, if the mass isn't still, its vector position will change with the time.

Let us now consider a generic trajectory of a mass in order to investigate its motion between two instants of time t and $t + \Delta t$. The position vectors that locate the mass position in the two previous instants of time will be $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$ respectively. Therefore the displacement of the mass in the interval of time Δt will be (see Fig. 4.3):

$$\Delta\mathbf{r}(t) = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \tag{4.3}$$

Taking into account the classical definition of velocity we could define the mean velocity in the finite interval of time Δt by means of the relation:

$$\overline{\mathbf{v}}(t) = \frac{\Delta\mathbf{r}(t)}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{4.4}$$

---

[19] The vectorial quantities will always be indicated by bold letters.

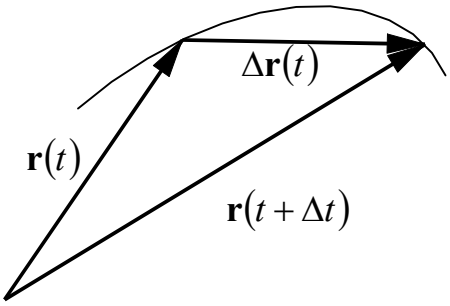

Figure 4.3: Displacement of a point in the interval of time Δt.

If we would the instantaneous velocity of the mass at the time t we should consider the limit of the mean velocity $\bar{\mathbf{v}}$ as $\Delta t \to 0$ [20]. Therefore the instantaneous velocity at time t will be defined as:

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \bar{\mathbf{v}}(t) \tag{4.5}$$

Remembering the definition of the position vector **r** we can write:

$$\begin{cases} \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \\ \mathbf{r}(t+\Delta t) = x(t+\Delta t)\mathbf{i} + y(t+\Delta t)\mathbf{j} \end{cases} \tag{4.6}$$

so that the mean velocity (4.4) can be written as:

$$\bar{\mathbf{v}}(t) = \frac{x(t+\Delta t) - x(t)}{\Delta t}\mathbf{i} + \frac{y(t+\Delta t) - y(t)}{\Delta t}\mathbf{j} \tag{4.7}$$

Substituting (4.7) in (4.5) we get:

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \bar{\mathbf{v}}(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}\,\mathbf{i} + \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}\,\mathbf{j} \tag{4.8}$$

[20] We will read this to mean: "Δt approaches zero".

Taking into account the definition of the derivative of a function we get the following expression of velocity:

$$\mathbf{v}(t)=\frac{d\,x(t)}{d\,t}\,\mathbf{i}+\frac{d\,y(t)}{d\,t}\,\mathbf{j} \tag{4.9}$$

or, better:

$$\mathbf{v}(t)=\dot{x}(t)\,\mathbf{i}+\dot{y}(t)\,\mathbf{j} \tag{4.10}$$

This is exactly the expression of the velocity that we will use in the formulation of the equations of motion for a model.
Obviously the velocity of the generic mass m_i is:

$$\mathbf{v}_i(t)=\dot{x}_i(t)\,\mathbf{i}+\dot{y}_i(t)\,\mathbf{j} \tag{4.11}$$

4.4 *THE ACCELERATION*

In addition to the velocity, it is possible to define the acceleration through the limit as $\Delta t\to 0$ of the mean acceleration $\overline{\mathbf{a}}(t)$:

$$\mathbf{a}(t)=\lim_{\Delta t\to 0}\overline{\mathbf{a}}(t)=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\mathbf{v}(\mathrm{t}+\Delta\mathrm{t})-\mathbf{v}(t)}{\Delta t} \tag{4.12}$$

Taking into account the expression (4.10) for the velocity, from (4.12) follows:

$$\mathbf{a}(t)=\lim_{\Delta t\to 0}\frac{\dot{x}(\mathrm{t}+\Delta\mathrm{t})-\dot{x}(t)}{\Delta t}\,\mathbf{i}+\lim_{\Delta t\to 0}\frac{\dot{y}(\mathrm{t}+\Delta\mathrm{t})-\dot{y}(t)}{\Delta t}\,\mathbf{j} \tag{4.13}$$

that is:

$$\mathbf{a}(t) = \ddot{x}(t)\ \mathbf{i} + \ddot{y}(t)\ \mathbf{j} \tag{4.14}$$

This is exactly the expression of the acceleration that we will use in the formulation of the equations of motion for a model.

For the generic mass m_i we also have:

$$\mathbf{a}_i(t) = \ddot{x}_i(t)\ \mathbf{i} + \ddot{y}_i(t)\ \mathbf{j} \tag{4.15}$$

5. TOPOLOGY OF A MODEL

The first question about this chapter could be: What exactly is a model? Obviously we need a definition. For this reason from now when we talk about a model we mean a system of free masses[21] themselves connected in various way by means of muscles and springs. And what about the topology? The topology of a model is simply the exact way in which the masses of a model are connected.

In order to define exactly the topology of a model let us consider a model made with *N* free masses. We will indicate with the integer number N_i the number of masses that are connected with the generic free mass m_i so that mass m_1 is connected with N_1 free masses, mass m_2 connected with N_2 masses, and so on. Obviously for a model made with *N* free masses we will have *N* integer numbers N_i. These parameters are very important even if they don't say anything about the exact connections between the masses. The only thing that we can determine with these parameters is that the mass m_i is connected with N_i other free masses; nothing more. We still don't know exactly which masses are connected to the mass m_i. For this reason we will introduce the *topological index* ε_{ij}.

The topological index ε_{ij} represents the index of the *j*-th mass connected with mass m_i. Obviously the index *i* and *j* will be such as:

[21] Fixed masses are not discussed in this article.

$$\begin{cases} i = 1,\ 2,\ 3,\ \ldots,\ N \\ j = 1,\ 2,\ \ldots,\ N_i \end{cases} \tag{5.1}$$

In order to well understand the exact definition of the topological index ε_{ij} let us consider the following example. In the model of Figure 5.1 we have 8 free masses.

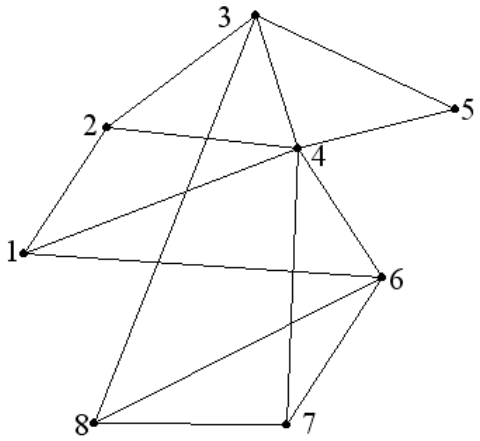

Figure 5.1: Example of a model

It is trivial to see that the mass m_1 is connected to 3 other masses, that mass m_2 is connected to 3 other masses, that mass m_3 is connected to 4 other masses, and so on. By using the definition of the parameter N_i we can simply write:

$$\begin{aligned} &N_1 = 3; \quad N_2 = 3; \\ &N_3 = 4; \quad N_4 = 6; \\ &N_5 = 2; \quad N_6 = 4; \\ &N_7 = 3; \quad N_8 = 3. \end{aligned}$$

But how are these masses connected? It is also trivial to see that the mass m_1 is connected to masses m_2, m_4 and m_6, mass m_2 to masses m_1, m_3 and m_4, and so on. Well, the values of the topological index tell us exactly the way in which the masses are connected. In the present case the values of the topological index are:

$$\varepsilon_{11}=2;\quad \varepsilon_{12}=4;\quad \varepsilon_{13}=6;$$
$$\varepsilon_{21}=1;\quad \varepsilon_{22}=3;\quad \varepsilon_{23}=4;$$
$$\varepsilon_{31}=2;\quad \varepsilon_{32}=4;\quad \varepsilon_{33}=5;\quad \varepsilon_{34}=8;$$
$$\varepsilon_{41}=1;\quad \varepsilon_{42}=2;\quad \varepsilon_{43}=3;\quad \varepsilon_{44}=5;\quad \varepsilon_{45}=6;\quad \varepsilon_{46}=7;$$
$$\varepsilon_{51}=3;\quad \varepsilon_{52}=4;$$
$$\varepsilon_{61}=1;\quad \varepsilon_{62}=4;\quad \varepsilon_{63}=7;\quad \varepsilon_{64}=8;$$
$$\varepsilon_{71}=4;\quad \varepsilon_{72}=6;\quad \varepsilon_{73}=8;$$
$$\varepsilon_{81}=3;\quad \varepsilon_{82}=6;\quad \varepsilon_{83}=7.$$

The topological index will be very important when we need to sum the elastic forces acting on a free mass in order to formulate the equations of motion for a model.

6. FORCES ANALYSIS

The movement of a mass is strictly related to the forces that are applied to the mass. For this reason, in this chapter we will talk about the different kinds of forces that a mass might encounter in its motion. There are essentially three such forces: the gravity forces, the damping forces (the friction), and the elastic forces.

6.1 *THE GRAVITY FORCES*

Regarding the gravity forces, to tell the truth, there isn't much to say. Each free mass will be subjected to a weight force that is proportional to the mass value by means of the constant *g*. In the applet, all free masses are the same, so each mass will be subjected to the force:

$$\mathbf{F}_i^g = -m_i\, g\, \mathbf{j} = -m\, g\, \mathbf{j} \qquad i = 1,\ 2,\ \ldots,\ N \tag{6.1}$$

where the symbol *m* has been assumed for the common value of all the masses. Obviously, in order to express the weight force, we need the unit vector **j** of the *y*-axis. Finally the minus sign in (6.1) means that gravity pushes the masses down. When the gravity reverse option in the applet is activated the expression (6.1) simply becomes:

$$\mathbf{F}_i^g = m_i\, g\, \mathbf{j} = m\, g\, \mathbf{j} \qquad i = 1,\ 2,\ \ldots,\ N \tag{6.2}$$

6.2 *THE DAMPING FORCES*

We already talked about the damping forces when we studied the free damped vibrations of a spring-mass system in order to calculate experimentally the damping constant of the applet. We have already said that a mass in motion in a viscous fluid with velocity **v** is subjected to a force **F** equal to (see Fig. 3.8):

$$\mathbf{F} = -f\, \mathbf{v} \tag{6.3}$$

Now, thanks to what we have already said in sections 3.4 and 4.3, we can write a more appropriate expression for the damping force acting on a generic mass m_i:

$$\mathbf{F}_i^f(t) = -f\,\mathbf{v}_i(t) \qquad i = 1,\ 2,\ \ldots,\ N \tag{6.4}$$

from which, thanks to the expression of velocity (4.11), follows:

$$\mathbf{F}_i^f(t) = -f\left[\dot{x}_i(t)\mathbf{i} + \dot{y}_i(t)\mathbf{j}\right] \qquad i = 1,\ 2,\ \ldots,\ N \tag{6.5}$$

Finally, taking into account the first expression of (3.47), we will write:

$$\mathbf{F}_i^f(t) = -m \cdot \bar{f} \cdot \left[\dot{x}_i(t)\mathbf{i} + \dot{y}_i(t)\mathbf{j}\right] \qquad i = 1,\ 2,\ \ldots,\ N \tag{6.6}$$

in which has been introduced the specific damping constant $\bar{f}$.

6.3 *THE ELASTIC FORCES*

The discussion of the elastic forces acting on a specific mass of a model is a little bit more complex, essentially for two reasons. The first is that each mass of a model is generally connected to more masses, so that often many elastic forces act upon a single mass. The second is that the elastic forces acting on a mass can come from springs or muscles. So, in order to find the exact expression for the resultant of many elastic forces acting on a single generic mass, we must first study springs and muscles. We will do this in the next chapter, so here we will talk about the elastic forces in general.

We have already talked about the elastic forces of a spring in section 3.2. In that section we said that a spring of original length l_0 extended until the length l ($l > l_0$) needs a traction force of intensity *F* proportional to the extension $\Delta l = l - l_0$ by means of the stiffness constant *k* (see Fig. 3.5):

$$F = k \cdot \Delta l \tag{6.7}$$

This is valid, obviously, from the spring's point of view. But now we are investigating the forces acting on the free masses by means of springs. These forces, thanks to Newton's third law[22], are equal in intensity to the forces acting on the springs but with opposite direction. Therefore, when a spring is extended, the forces act on the free masses at its extremities pushing them toward each other. Instead, when a spring is compressed, the forces act on the free masses at its extremities pushing them away from each other (see Fig. 6.1).

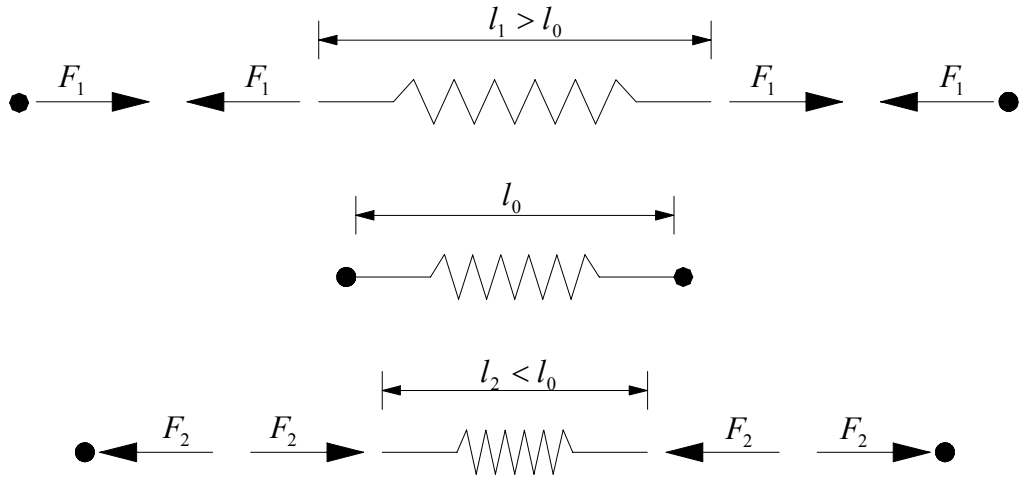

Figure 6.1: Forces acting on the masses at the end of a spring.

7. SPRINGS AND MUSCLES

In order to get the exact expression for the resultant of elastic forces acting on a generic mass m_i, we must first investigate springs and muscles. Without any delay, let us begin these investigations.

7.1 *THE SPRINGS*

Let us consider a spring that connects two free masses, m_i and m_j, in a hypothetical model. We will indicate with l_{ij}^r the length at rest of the spring that connects masses *i* and *j*. This length is the length assigned to the spring when the two masses are originally connected

[22] The principle of action and reaction.

in the applet's construct mode. This value will never change during the simulation, and is important in order to evaluate the elastic forces acting on the masses m_i and m_j.

We will indicate with $l_{ij}(t)$ the effective length of a spring during the simulation. Obviously this length will change continuously so it will be a function of time. Its value is determined by the coordinates of its extremities; i.e. its value depends on the coordinates of the free masses m_i and m_j. Therefore, applying the well-known Theorem of Pythagoras we have (see Fig. 7.1):

$$l_{ij}(t) = \left\{ \left[x_i(t) - x_j(t) \right]^2 + \left[y_i(t) - y_j(t) \right]^2 \right\}^{\frac{1}{2}} \tag{7.1}$$

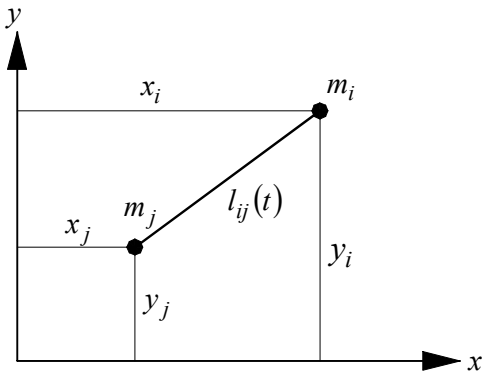

Figure 7.1: Length of a spring.

The stretch of the spring *ij* in the generic instant of time *t* is:

$$\Delta l_{ij}(t) = l_{ij}(t) - l_{ij}^r \tag{7.2}$$

Therefore, when $\Delta l_{ij}(t) > 0$, the spring is in traction, while when $\Delta l_{ij}(t) < 0$, the spring is in compression. As we discussed in the previous chapter, when the spring *ij* is in traction, the elastic forces acting on masses *i* and *j* will push them together, and when the spring *ij* is in compression, the elastic forces acting on masses *i* and *j* will push them apart. The direction of these forces, whatever the sign of $\Delta l_{ij}(t)$ may be, will correspond with the direction of the

straight line on which lies the spring *ij* while the intensity of the two forces will be proportional to the stretch $\Delta l_{ij}(t)$ by means of the well-known stiffness *k*.

Let us now concentrate our attention on the elastic force acting on the mass m_i. We will represent this force with the symbol $\mathbf{F}_{ij}^{e}(t)$ (elastic force acting on the mass *i* by means of the spring that connects the masses *i* and *j*). Obviously, $\mathbf{F}_{ji}^{e}(t) = -\mathbf{F}_{ij}^{e}(t)$.

Regarding the intensity of these two forces, we can immediately write that:

$$\left|\mathbf{F}_{ij}^{e}(t)\right| = \left|\mathbf{F}_{ji}^{e}(t)\right| = k \cdot \left|\Delta l_{ij}(t)\right| \tag{7.3}$$

in which we have indicated with the symbol $|\mathbf{F}|$ the intensity of the force **F**, while we have indicated with the symbol $\left|\Delta l_{ij}(t)\right|$ the absolute value of the stretch $\Delta l_{ij}(t)$.

In order to fully characterize the elastic force $\mathbf{F}_{ij}^{e}(t)$, we need to know its direction. As we have said before, the direction of these forces corresponds with the direction of the straight line on which lies the spring *ij*, so the direction of the force $\mathbf{F}_{ij}^{e}(t)$ will be the same or opposite the direction of the vector:

$$\mathbf{r}_{ij}(t) = \mathbf{r}_j(t) - \mathbf{r}_i(t) = \left[x_j(t) - x_i(t)\right]\mathbf{i} + \left[y_j(t) - y_i(t)\right]\mathbf{j} \tag{7.4}$$

This is the vector that joins the extremity of vector *i* to the extremity of the vector *j* (see Fig. 7.2). Obviously the length of this vector is the same as the length $l_{ij}(t)$:

$$\left|\mathbf{r}_{ij}(t)\right| = l_{ij}(t) \tag{7.5}$$

so that the unit vector of the direction *ij* is:

$$\hat{\mathbf{r}}_{ij}(t)=\frac{\mathbf{r}_{ij}(t)}{l_{ij}(t)}=\frac{\mathbf{r}_j(t)-\mathbf{r}_i(t)}{l_{ij}(t)}=\frac{\left[x_j(t)-x_i(t)\right]\mathbf{i}+\left[y_j(t)-y_i(t)\right]\mathbf{j}}{\sqrt{\left[x_j(t)-x_i(t)\right]^2+\left[y_j(t)-y_i(t)\right]^2}} \tag{7.6}$$

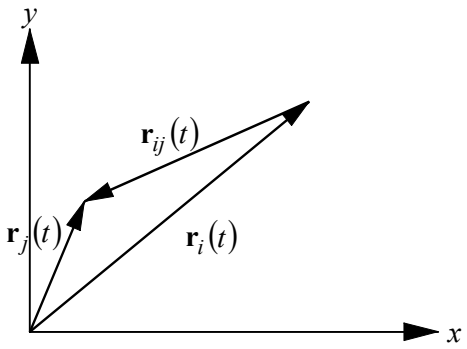

Figure 7.2: The vector $\mathbf{r}_{ij}(t)$

The unit vector $\hat{\mathbf{r}}_{ij}(t)$, like the vector $\mathbf{r}_{ij}(t)$, is directed from *i* to *j*. This is the same as the direction of the elastic force $\mathbf{F}_{ij}^{e}(t)$ when the stretch $\Delta l_{ij}(t)$ is positive. Therefore we can write a general expression of the force $\mathbf{F}_{ij}^{e}(t)$ simply by multiplying $k\cdot\Delta l_{ij}(t)$ with $\hat{\mathbf{r}}_{ij}(t)$:

$$\mathbf{F}_{ij}^{e}(t)=k\cdot\Delta l_{ij}(t)\hat{\mathbf{r}}_{ij}(t) \tag{7.7}$$

This expression is valid in general. Indeed, if the stretch is positive, then as we have stated before, the force acting on the mass *i* will directed from *i* to *j* like the unit vector $\hat{\mathbf{r}}_{ij}(t)$, while, if the stretch is negative, the force acting on the mass *i* will be directed from *j* to *i,* like the direction of the vector obtained by multiplying a negative quantity ($\Delta l_{ij}(t)$) with the unit vector $\hat{\mathbf{r}}_{ij}(t)$.

Using the previous expressions (7.1), (7.2) and (7.6), from (7.7) we get a more explicit expression for the elastic force $\mathbf{F}_{ij}^{e}(t)$:

$$\mathbf{F}_{ij}^{e}(t)=k\left[l_{ij}(t)-l_{ij}^{r}\right]\frac{\mathbf{r}_{ij}(t)}{l_{ij}(t)} \tag{7.8}$$

where:

$$\begin{cases} l_{ij}(t) = \sqrt{\left[x_j(t) - x_i(t)\right]^2 + \left[y_j(t) - y_i(t)\right]^2} \\ \mathbf{r}_{ij}(t) = \left[x_j(t) - x_i(t)\right]\mathbf{i} + \left[y_j(t) - y_i(t)\right]\mathbf{j} \end{cases} \tag{7.9}$$

7.2 *THE MUSCLES*

We all know what a spring is. We have all seen real springs in our own lives. But what exactly is a muscle? In the sodaconstructor, muscles are simply springs with the length at rest l_{ij}^r changing in time with harmonic law. Certainly, this sort of object doesn't exist in the real world. In reality a spring's length at rest doesn't change over time. Muscles are simply abstract objects invented by Ed Burton (I suppose) in order to give energy to our models. You could say that the sodaconstructor muscle is a simplified version of the human muscle.
In order to define mathematically these new entities, let us consider two free masses m_i and m_j connected by a muscle. As stated before, the length at rest of the muscle is:

$$l_{ij}^r(t) = l_{ij}^0\left[1 + \alpha_{ij} \cdot \beta \cdot \sin\left(\omega t + \varphi_{ij}\right)\right] \tag{7.10}$$

where a lot of new parameters have been introduced. The parameters in (7.10) that have the index *ij* are parameters closely related to the specific muscle that joins the masses *i* and *j*. These parameters (l_{ij}^0, α_{ij}, φ_{ij}) can be different for each muscle of a model. The other parameters (β, ω) are valid for all the muscles of a model.

The first and most intuitive parameter that has been introduced in (7.10) is the length l_{ij}^0. This length is the length assigned to the muscle when the two masses are originally connected in the applet's construct mode. If this were a spring and not a muscle, the length l_{ij}^0 would simply be the length at rest l_{ij}^r.

In order to explain all the other new parameters in (7.10) we will refer to the sodaconstructor's muscle control panel, as shown in the following Figure 7.3:

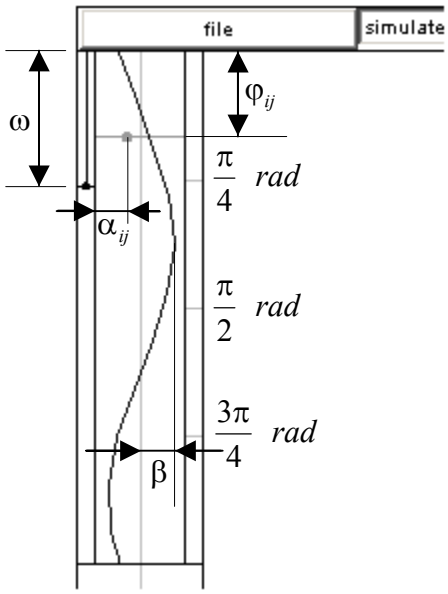

Figure 7.3: muscle control panel.

Let us go in order:

- ω is the parameter that determines the speed of the oscillation for all the muscles in a model. Its value is expressed in rad/frm and it can be changed with the cursor shown in Figure 7.3. When this cursor is placed on the top, as everyone knows, the value of the parameter ω is zero. We could define ω in the following various ways: *pulsation*, *frequency*, *circular frequency*, etc.
- φ_{ij} is the parameter of phase for the specific muscle *ij*. By adjusting this parameter we set the delay before muscle *ij* reaches its maximum amplitude. In Figure 7.3 it is possible to see the value of the phase φ_{ij} in the muscles control panel. When the phase cursor is on the top the phase is equal to $\varphi_{ij} = 0\ rad$. The intermediate positions correspond respectively the values of $\pi/4\ rad$, $\pi/2\ rad$ and $3\pi/4\ rad$. Thanks to this parameter, the model constructors are able to synchronize harmoniously the movement of their precious and spectacular models ;)

- β is the parameter that fixes the wave amplitude. This parameter can assume any value in the interval $[0,1]$. When the sinusoidal wave is simply a straight line, the value of the parameter β is 0. When the wave has the maximum amplitude β is equal to 1 (see Fig. 7.3). Obviously, the value of β is the same for all the muscles of a model.
- α_{ij} is the parameter that determines the benefit of the wave amplitude β for the muscle *ij* (its relative amplitude). Just like the amplitude β, α_{ij} can assume all the values in the interval $[0,1]$. α_{ij} is equal to 0 when the position of the cursor (see Fig. 7.3) is completely on the left, while α_{ij} is equal to 1 when the cursor is completely on the right.

In order to better understand these parameters let us consider a few examples. The first example that I want to take into consideration is the case of a free muscle with $\alpha_{ij} = \beta = 1$. In this case, (7.10) becomes:

$$l_{ij}^{r}(t) = l_{ij}^{0}\left[1 + \sin\left(\omega t + \varphi_{ij}\right)\right] \tag{7.11}$$

which shows that the length of the free muscle will change periodically from 0 to $2 l_{ij}^{0}$.

The second example that I want to take into consideration is the case of a muscle with $\alpha_{ij} = 0$. In this case, no matter what the value of the amplitude, from (7.11) simply follows:

$$l_{ij}^{r}(t) = l_{ij}^{0} \tag{7.12}$$

In this case the muscle becomes a spring. For this reason, it is absolutely valid to think of all springs as muscles with the parameter $\alpha_{ij} = 0$.

What about the forces acting on the masses at the extremities of a muscle? The expression of these forces is exactly the same as it is with the elastic forces due to springs. The only difference is in the expression of the length l_{ij}^{r}: for springs this length is constant, while for muscles this length changes over time with the law (7.10).

Therefore, the most general expression of the elastic force acting on a generic mass *i* by means of a muscle or spring that connects mass *i* to mass *j* is:

$$\mathbf{F}_{ij}^{e}(t)=k\left[l_{ij}(t)-l_{ij}^{r}(t)\right]\frac{\mathbf{r}_{ij}(t)}{l_{ij}(t)} \tag{7.13}$$

in which:

$$\begin{cases} l_{ij}(t)=\sqrt{\left[x_j(t)-x_i(t)\right]^2+\left[y_j(t)-y_i(t)\right]^2} \\ l_{ij}^{r}(t)=l_{ij}^{0}\left[1+\alpha_{ij}\cdot\beta\cdot\sin\left(\omega t+\varphi_{ij}\right)\right] \\ \mathbf{r}_{ij}(t)=\left[x_j(t)-x_i(t)\right]\mathbf{i}+\left[y_j(t)-y_i(t)\right]\mathbf{j} \end{cases} \tag{7.14}$$

7.3 *RESULTANT OF THE ELASTIC FORCES ON A FREE MASS*

At this point we are able to define the exact expression of the resultant of the elastic forces acting on a free mass of a model.

Remembering the definitions of the integer parameters N_i and the topological index ε_{ij} we can write the resultant of the elastic forces acting on the generic mass m_i by means of the following relation:

$$\mathbf{F}_{i}^{e}(t)=\sum_{j=1}^{N_i}\mathbf{F}_{i\varepsilon_{ij}}^{e}(t)=\sum_{j=1}^{N_i}k\left[l_{i\varepsilon_{ij}}(t)-l_{i\varepsilon_{ij}}^{r}(t)\right]\frac{\mathbf{r}_{i\varepsilon_{ij}}(t)}{l_{i\varepsilon_{ij}}(t)} \tag{7.15}$$

In order to formulate the equations of motion of a model, it will be convenient to write (7.15) taking into account the expression (3.17) of the stiffness *k*:

$$\mathbf{F}_{i}^{e}(t)=\sum_{j=1}^{N_i}\mathbf{F}_{i\varepsilon_{ij}}^{e}(t)=m\cdot\bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t)-l_{i\varepsilon_{ij}}^{r}(t)\right]\frac{\mathbf{r}_{i\varepsilon_{ij}}(t)}{l_{i\varepsilon_{ij}}(t)} \tag{7.16}$$

There is one last question about the mathematical expression of elastic forces. What happens when two masses are in same place on the screen? This event, even if it isn't realistic, physically speaking, could happen[23]. In this case the expression (7.8) becomes indeterminate (mathematically we have met an indetermination of the kind $0/0$). The soda algorithm must have some control in order to avoid an error like the *division by zero error*. I've verified that this control effectively exists through a demonstration in the following Example in which two masses are perfectly overlapped. The applet simply fails to take into consideration the elastic forces between masses that are perfectly overlapped.

[23] In linear motors, for example, this happens systematically.

8 EQUATIONS OF THE MOTION OF A MODEL

We now have all the expressions of the forces that could act on a generic mass m_i. These forces are:

- *The gravity force:*

$$\mathbf{F}_i^g = -m\,g\,\mathbf{j} \tag{8.1}$$

- *The damping force:*

$$\mathbf{F}_i^f(t) = -m\cdot\bar{f}\cdot\left[\dot{x}_i(t)\mathbf{i} + \dot{y}_i(t)\mathbf{j}\right] \tag{8.2}$$

- *The elastic force:*

$$\mathbf{F}_i^e(t) = m\cdot\bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l_{i\varepsilon_{ij}}^r(t)\right]\frac{\left[x_{\varepsilon_{ij}}(t) - x_i(t)\right]\mathbf{i} + \left[y_{\varepsilon_{ij}}(t) - y_i(t)\right]\mathbf{j}}{l_{i\varepsilon_{ij}}(t)} \tag{8.3}$$

Therefore their resultant $\mathbf{F}_i(t)$ will be:

$$\mathbf{F}_i(t) = \mathbf{F}_i^g + \mathbf{F}_i^f(t) + \mathbf{F}_i^k(t) \tag{8.4}$$

Combining the two directional components *x* and *y,* the resultant $\mathbf{F}_i(t)$ can be written as:

$$\mathbf{F}_i(t) = F_i^x(t)\mathbf{i} + F_i^y(t)\mathbf{j} \tag{8.5}$$

where the expressions of the components $F_i^x(t)$ and $F_i^y(t)$, according to (8.1,2,3,4), can be written as:

$$\begin{cases} F_i^x(t) = m\left\{-\bar{f}\,\dot{x}_i(t) + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^r_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[x_{\varepsilon_{ij}}(t) - x_i(t)\right]}{l_{i\varepsilon_{ij}}(t)}\right\} \\ F_i^y(t) = m\left\{-g - \bar{f}\,\dot{y}_i(t) + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^r_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[y_{\varepsilon_{ij}}(t) - y_i(t)\right]}{l_{i\varepsilon_{ij}}(t)}\right\} \end{cases} \tag{8.6}$$

In order to get the system of differential equations that governs the motion of a model, we will apply Newton's second law to all the *N* masses of the model. Therefore, we will have:

$$\mathbf{F}_i(t) = m\,\mathbf{a}_i(t) \qquad i = 1,\ 2,\ \ldots,\ N \tag{8.7}$$

or, taking into account the expressions (4.15) and (8.5), also:

$$F_i^x(t)\mathbf{i} + F_i^y(t)\mathbf{j} = m\left[\ddot{x}_i(t)\mathbf{i} + \ddot{y}_i(t)\mathbf{j}\right] \qquad i = 1,\ 2,\ \ldots,\ N \tag{8.8}$$

Assembling together the components of *x* and *y* of all these equations and taking into account expressions (8.6) of the components $F_i^x(t)$ and $F_i^y(t)$ we will get:

$$\begin{cases} m\,\ddot{x}_i(t) = m\left\{-\bar{f}\,\dot{x}_i(t) + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^r_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[x_{\varepsilon_{ij}}(t) - x_i(t)\right]}{l_{i\varepsilon_{ij}}(t)}\right\} \\ m\,\ddot{y}_i(t) = m\left\{-g - \bar{f}\,\dot{y}_i(t) + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^r_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[y_{\varepsilon_{ij}}(t) - y_i(t)\right]}{l_{i\varepsilon_{ij}}(t)}\right\} \end{cases} \qquad i = 1,\ 2,\ \ldots,\ N \tag{8.9}$$

At this point it is possible to definitively solve the mystery of the value of the mass *m*. As stated in Chapter 2, it isn't necessary to assume any specific value for the mass *m*, consistent to all the free masses in the applet. Indeed, whatever the value of *m* might be, in (8.9) it can be seen that the constant *m* can be simplified everywhere. Obviously this is possible only because all masses in the applet are the same[24].

[24] i.e. all the masses have the same inertial properties.

The equations (8.9) can more conveniently be written as:

$$\begin{cases} \ddot{x}_i(t) + \bar{f}\,\dot{x}_i(t) = f_i^{\,x}(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, t) \\ \ddot{y}_i(t) + \bar{f}\,\dot{y}_i(t) = f_i^{\,y}(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, t) \end{cases} \qquad i = 1,\ 2,\ \ldots,\ N \tag{8.10}$$

in which the known scalar functions $f_i^{\,x}$ and $f_i^{\,y}$ have the expressions:

$$\begin{cases} f_i^{\,x}(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, t) = \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^{\,r}_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[x_{\varepsilon_{ij}}(t) - x_i(t)\right]}{l_{i\varepsilon_{ij}}(t)} \\ f_i^{\,y}(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N, t) = -g + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t) - l^{\,r}_{i\varepsilon_{ij}}(t)\right]\dfrac{\left[y_{\varepsilon_{ij}}(t) - y_i(t)\right]}{l_{i\varepsilon_{ij}}(t)} \end{cases} \tag{8.11}$$

The equations (8.10) represent a system of $2N$ non-linear[25] differential equations of second order in $2N$ unknown functions represented by the coordinates of the model's free masses $x_i(t), y_i(t)\ [i = 1, 2, \ldots, N]$. Thanks to the numerical resolution[26] of this system of equations we can determine the movement of our model over time.

9. NUMERICAL RESOLUTION OF THE EQUATIONS OF MOTION

In the past, even though many problems were already formulated in terms of differential equations, their exact resolutions[27] weren't always obtainable. Only in a few cases was it possible to obtain an exact solution; in all the remaining cases almost nothing could be done. Sometimes it was possible to use approximate methods of resolution, but in these methods a lot of calculus had to be done. Today, thanks to our beloved computers, this isn't a problem anymore. Today we have at our disposal a large amount of numerical techniques that allow us to get a resolution of any differential problem with the accuracy wanted. Finding a solution to a problem like those defined in equations (8.10) and (8.11) only thirtyfive years ago would

[25] The functions $f_i^{\,x}$ and $f_i^{\,y}$ are non-linear functions.

[26] Unfortunately, the resolution (integration) of the equations (8.11) is possible only in a numerical way.

[27] i.e. the resolution in terms of well-known mathematical functions.

have required the computers of NASA. Today, there are a variety of pocket videogames that are able to solve problems more complex than ours. Today it might also be accurate that the true power of a nation is in the capacity of her computers to go fast (capacity expressed in billions of calculus per second).

Here I will show you the simplest numerical technique that we need in order to solve the equations of motion (8.10). This numerical technique, usually called the *finite differences*, as well as all the other more sophisticated numerical techniques, will transform the system of differential equations (8.10) into a system of algebraic equations that, step by step, allow us to get the positions of the masses of a model over time.

In order to pursue this objective we absolutely need some approximated expressions for the scalar velocities $\dot{x}_i(t)$, $\dot{y}_i(t)$ and the scalar accelerations $\ddot{x}_i(t)$, $\ddot{y}_i(t)$. Therefore we will start looking for an approximated expression for the first derivative of a generic real function $f(x)$. We know that the exact expression of the first derivative of a function $f(x)$ is defined by the limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{9.1}$$

Looking at this definition it is immediately possible to understand that an approximate value for the first derivative $f'(x)$ could be calculated choosing a very small value of the finite interval Δx. Therefore the expression:

$$f'(x) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{9.2}$$

will be more accurate as Δx gets smaller, so that we will use this expression in order to approximate the first derivative of a function.

What about the second derivative $f''(x)$? By definition we can write:

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} \tag{9.3}$$

so that, just like the first derivative, the approximate expression for the second derivative will be:

$$f''(x) \cong \frac{f'(x+\Delta x) - f'(x)}{\Delta x} \tag{9.4}$$

This expression isn't particularly useful because it contains the first derivative of the function $f(x)$. Therefore, since from expression (9.2) follows:

$$f'(x+\Delta x) \cong \frac{f(x+2\Delta x) - f(x+\Delta x)}{\Delta x} \tag{9.5}$$

substituting (9.2) and (9.5) in the approximate expression of the second derivative (9.4) we get:

$$f''(x) \cong \frac{f(x+2\Delta x) - 2f(x+\Delta x) + f(x)}{\Delta x^2} \tag{9.6}$$

In order to evaluate the approximate expressions of the first and second derivatives of the unknown function of our problem we will use respectively the relations (9.2) and (9.6). We will have:

$$\begin{cases} \dot{x}_i(t) \cong \dfrac{x_i(t+\Delta t) - x_i(t)}{\Delta t} \\ \dot{y}_i(t) \cong \dfrac{y_i(t+\Delta t) - y_i(t)}{\Delta t} \\ \ddot{x}_i(t) \cong \dfrac{x_i(t+2\Delta t) - 2x_i(t+\Delta t) + x_i(t)}{\Delta t^2} \\ \ddot{y}_i(t) \cong \dfrac{y_i(t+2\Delta t) - 2y_i(t+\Delta t) + y_i(t)}{\Delta t^2} \end{cases} \tag{9.7}$$

These expressions will be more accurate as the finite interval of time Δt is smaller.

Using the approximate expression (9.7) in the equations of motion (8.10) we get:

$$\begin{cases} \dfrac{x_i(t+2\Delta t)-2x_i(t+\Delta t)+x_i(t)}{\Delta t^2}+\bar{f}\,\dfrac{x_i(t+\Delta t)-x_i(t)}{\Delta t}=f_i^{\,x}(t) \\ \dfrac{y_i(t+2\Delta t)-2y_i(t+\Delta t)+y_i(t)}{\Delta t^2}+\bar{f}\,\dfrac{y_i(t+\Delta t)-y_i(t)}{\Delta t}=f_i^{\,y}(t) \end{cases} \qquad i=1,\,2,\,\ldots,\,N \quad (9.8)$$

from which follows:

$$\begin{cases} x_i(t+2\Delta t)=(2-\bar{f}\,\Delta t)x_i(t+\Delta t)+(\bar{f}\,\Delta t-1)x_i(t)+f_i^{\,x}(t)\Delta t^2 \\ y_i(t+2\Delta t)=(2-\bar{f}\,\Delta t)y_i(t+\Delta t)+(\bar{f}\,\Delta t-1)y_i(t)+f_i^{\,y}(t)\Delta t^2 \end{cases} \qquad i=1,2,\ldots,N \quad (9.9)$$

These expressions are very important in finding the approximate solutions of the equations (8.10) because they allow us to find the position of all *N* masses of a model in the instant of time $t+2\Delta t$, while previously we knew the position of all *N* masses in the instants of time $t+\Delta t$ and *t*. This means that if we know the positions and velocities of all the masses in a particular instant of time, using the relations (9.9) iteratively we will be able to get the evolution of the position of all the masses over time; i.e. the movement of our model. The positions and the velocities of all the masses in a particular starting instant are called initial conditions, and they are fundamental in order to solve numerically our differential equations.
Above I said that using the equations (9.9) iteratively it is possible to calculate the positions of the masses of a model over time. But what exactly does it mean to use the equations iteratively? In order to understand this let us introduce the following position:

$$\begin{cases} t_j=t_{j-1}+\Delta t \\ x_i(t_j)=x_i^j \\ y_i(t_j)=y_i^j \\ f_i^{\,x}(t_j)=f_{ij}^{\,x} \\ f_i^{\,y}(t_j)=f_{ij}^{\,y} \end{cases} \qquad j=1,2,3,\ldots,\infty \quad (9.10)$$

If we choose as the initial instant the generic time t_k, from the first of (9.10) follows:

$$\begin{cases} t_{k+1} = t_k + \Delta t \\ t_{k+2} = t_{k+1} + \Delta t = t_k + 2\Delta t \end{cases} \tag{9.11}$$

from which it is possible to see that the three instants of time t_k, t_{k+1} and t_{k+2} are in a series like the three instants of time t, $t + \Delta t$ and $t + 2\Delta t$ from (9.9). Therefore, these relations can also be written as:

$$\begin{cases} x_i(t_{k+2}) = (2 - \bar{f}\,\Delta t)x_i(t_{k+1}) + (\bar{f}\,\Delta t - 1)x_i(t_k) + f_i^x(t_k)\Delta t^2 \\ y_i(t_{k+2}) = (2 - \bar{f}\,\Delta t)y_i(t_{k+1}) + (\bar{f}\,\Delta t - 1)y_i(t_k) + f_i^y(t_k)\Delta t^2 \end{cases} \qquad i = 1, 2, \ldots, N \tag{9.12}$$

or better:

$$\begin{cases} x_i^{k+2} = (2 - \bar{f}\,\Delta t)x_i^{k+1} + (\bar{f}\,\Delta t - 1)x_i^k + f_{ik}^x\,\Delta t^2 \\ y_i^{k+2} = (2 - \bar{f}\,\Delta t)y_i^{k+1} + (\bar{f}\,\Delta t - 1)y_i^k + f_{ik}^y\,\Delta t^2 \end{cases} \qquad i = 1, 2, \ldots, N \tag{9.13}$$

from which it is possible to understand how, by simply knowing the positions of all the masses of a model in the two instants of time t_k and t_{k+1}, it is possible to get the positions of the same masses in the next instant of time t_{k+2}. Since the instant of time t_k is totally arbitrary, it is also possible to understand how the relations (9.13) are absolutely general. Therefore these relations can be used iteratively in order to find how the positions of the masses of a model change over time. Indeed, if we know, for example, the positions of the free masses of a model in the instants of time t_0 and t_1, thanks to (9.13) it is possible to find the position of the masses in the instant of time t_2. Then, applying (9.13) once again, it will be possible to find the position of all the masses at the instant of time t_3. In this way, we can find the positions of all the masses at the instants of time t_4, t_5, t_6 and so on indefinitely.

Before I close this chapter I will say one more thing. The expressions (9.13) might appear to be very simple and fast in application, but it is important to remember that the terms f_{ik}^x and

f_{ik}^{y} must be recalculated at every iteration. Their expressions can be obtained by simply going back to see all the positions previously assumed. These expressions are:

$$\begin{cases} f_{ik}^{x} = \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t_k) - l^{r}_{i\varepsilon_{ij}}(t_k) \right] \dfrac{\left[x_{\varepsilon_{ij}}(t_k) - x_i(t_k) \right]}{l_{i\varepsilon_{ij}}(t_k)} \\ f_{ik}^{y} = -g + \bar{k}\sum_{j=1}^{N_i}\left[l_{i\varepsilon_{ij}}(t_k) - l^{r}_{i\varepsilon_{ij}}(t_k) \right] \dfrac{\left[y_{\varepsilon_{ij}}(t_k) - y_i(t_k) \right]}{l_{i\varepsilon_{ij}}(t_k)} \end{cases} \tag{9.14}$$

where:

$$\begin{cases} l_{ij}(t) = \sqrt{\left[x_j(t) - x_i(t) \right]^2 + \left[y_j(t) - y_i(t) \right]^2} \\ l^{r}_{ij}(t) = l^{0}_{ij}\left[1 + \alpha_{ij} \cdot \beta \cdot \sin\left(\omega t + \varphi_{ij} \right) \right] \end{cases} \tag{9.15}$$

It is now perfectly clear that the main computational charge is due to the determination of these parameters at each cycle of iteration of (9.13).

What about the interval of time Δt? The choice of this value (usually a small fraction of frame) must be done with much caution. Indeed, if Δt is very small, the simulation will be pretty slow. However, if Δt is big, the solution that we get by applying the above equations will be completely wrong. It is also important to say that the choice of Δt also depends on the numerical technique chosen. Here I've talked about the finite differences method, which is the simplest numerical technique but also the least accurate. In the sodaconstructor applet, the numerical technique used is *Euler's Method*. I'll talk about this and other numerical techniques in Appendix C.

10. THE STATIC AND QUASI-STATIC SIMPLIFIED PROBLEM

The previous equations of motions (8.9) were obtained simply by applying Newton's second law to each mass of a model. The terms strictly related to the motion of free masses arise in these equations. These terms are the damping forces and the inertial forces[28]. These terms are respectively related to the velocities and the accelerations of the free masses. In many cases, however, the movement of the model could be so slow that these forces become themselves negligible. In these cases the only forces acting on a model are the elastic forces[29]. To study the movement of the model in these situations, it is only necessary to consider the elastic forces, like in the static case. Therefore, when we study a model without taking into account inertial forces or damping forces, we will say that we are studying a quasi-static model.

In this chapter, we will be able to solve many practical problems simply by not taking these forces into account.

10.1 *SYSTEM OF EQUATIONS*

By ignoring the damping and inertial forces, from the equations (8.9) follows this system of equations:

$$\begin{cases} \sum_{j=1}^{N_i} \left(l_{i\varepsilon_{ij}} - l^r_{i\varepsilon_{ij}} \right) \dfrac{x_{\varepsilon_{ij}} - x_i}{l_{i\varepsilon_{ij}}} = 0 \\ \sum_{j=1}^{N_i} \left(l_{i\varepsilon_{ij}} - l^r_{i\varepsilon_{ij}} \right) \dfrac{y_{\varepsilon_{ij}} - y_i}{l_{i\varepsilon_{ij}}} = 0 \end{cases} \quad i = 1,\ 2,\ \ldots,\ N \tag{10.1}$$

in which the time hasn't even been taken into account. This is because even if we have muscles in the model, the solution of the problem would be determined instant after instant. Therefore, in principle, there is no difference between the static case and the quasi-static case. In any case, we will only consider springs.

The system of equations (10.1) can be more conveniently written as:

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[28] in the sense of d'Alembert's principle.

[29] and, eventually, the gravity forces.

$$\begin{cases} \sum_{j=1}^{N_i} \left(1 - \frac{l^r_{i\varepsilon_{ij}}}{l_{i\varepsilon_{ij}}} \right) \left(x_{\varepsilon_{ij}} - x_i \right) = 0 \\ \sum_{j=1}^{N_i} \left(1 - \frac{l^r_{i\varepsilon_{ij}}}{l_{i\varepsilon_{ij}}} \right) \left(y_{\varepsilon_{ij}} - y_i \right) = 0 \end{cases} \qquad i = 1,\ 2,\ \ldots,\ N \tag{10.2}$$

where:

$$l_{i\varepsilon_{ij}} = \sqrt{\left(x_{\varepsilon_{ij}} - x_i \right)^2 + \left(y_{\varepsilon_{ij}} - y_i \right)^2} \tag{10.3}$$

In this system of equations the data are the lengths at rest of all the springs in the model, while the unknown quantities are the coordinates of the free masses in the equilibrium configuration.

Before we examine some applications of quasi-static problems, it is important to explain a few things about the system of equations (10.2). First, this system of equations isn't linear. This fact could be a big problem, because there is no general way to solve non-linear systems of equations. Some solutions can be obtained in particular cases, but in all the remaining cases the only way to solve the system of equations is to use numerical algorithms. The simplest numerical algorithm for the resolution of non-linear systems of equations is the generalization for more than one non-linear equation of Newton's algorithm[30]. Initially, I planned to add a short paragraph about this method, but for now I have changed my mind. The main reason is that we will not really need to solve the (10.2) system of equations in order to study practical problems in the ordinary life of a typical sodaconstructorman/woman. As we will see, it is possible to solve static and quasi-static problems without even considering this system of equations. Therefore, if anyone would like to learn about these numerical techniques, feel free to contact me and I will be happy to give you some explanation and/or reference.

The second thing I would like to say about the (10.2) system of equations is that its solution isn't unique. The same initial problem, even without taking into account the rigid translations

[30] The method of the tangents for a non-linear equation.

of a model, generally has more than one solution. Moreover, the number of solutions is proportionate to the complexity of the model. In order to demonstrate this characteristic that is common to most nonlinear systems of equations, it isn't necessary to trouble mathematics. Rather, it will be sufficient to just think about how many models are so delicate that their shape changes irremediably and definitively with just a little touch of the mouse.
In the rest of this chapter I will take into consideration some typical structures used by sodapeople in order to get mathematical explanations of their static shapes.

10.2 *REGULAR POLYGONS*

In this simple case, it is easy to get a solution for the system of equations (10.2). Actually, this is the only case in which we will bother with the previous system. The other problems will be solved using the simplified hypothesis of the quasi-static problems.
Let us consider the hexagonal model of Figure 10.1:

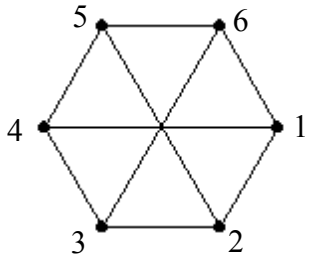

Figure 10.1: Hexagonal model.

This hexagonal model has been realized using zipsprings on the border and springs of finite length inside the model. The lengths at rest in this model are the data of the problem. They are:

$$\begin{cases} l^r_{12} = l^r_{23} = l^r_{34} = l^r_{45} = l^r_{56} = l^r_{61} = 0 \ \ pxl \\ l^r_{14} = l^r_{25} = l^r_{36} = l \end{cases} \tag{10.4}$$

where l has a particular value expressed in *pxl* that in this case isn't important to define exactly. Here, it is good enough to know that its value is the data of the problem. The unknown quantities of our problem are the coordinates of the free masses in the static

equilibrium configuration. Therefore, in order to make the system (10.2) explicit for the problem that we are studying, it is important to get the values of the topological index. Looking the model of Figure 10.1 we will find:

$$\begin{cases} N=6 \\ N_1=3 \quad \varepsilon_{11}=2 \quad \varepsilon_{12}=4 \quad \varepsilon_{13}=6 \\ N_2=3 \quad \varepsilon_{21}=1 \quad \varepsilon_{22}=3 \quad \varepsilon_{23}=5 \\ N_3=3 \quad \varepsilon_{31}=2 \quad \varepsilon_{32}=4 \quad \varepsilon_{33}=6 \\ N_4=3 \quad \varepsilon_{41}=1 \quad \varepsilon_{42}=3 \quad \varepsilon_{43}=5 \\ N_5=3 \quad \varepsilon_{51}=2 \quad \varepsilon_{52}=4 \quad \varepsilon_{53}=6 \\ N_6=3 \quad \varepsilon_{61}=1 \quad \varepsilon_{62}=3 \quad \varepsilon_{63}=5 \end{cases} \tag{10.5}$$

Thanks to these values it is possible to make the system of equations (10.2) explicit. Let us first consider the equation that follows from (10.2) for the coordinates x when the index $i=1$. We will have:

$$\sum_{j=1}^{N_1}\left(1-\frac{l^r_{1\varepsilon_{1j}}}{l_{1\varepsilon_{1j}}}\right)\left(x_{\varepsilon_{1j}}-x_1\right)=0$$

or better:

$$\left(1-\frac{l^r_{1\varepsilon_{11}}}{l_{1\varepsilon_{11}}}\right)\left(x_{\varepsilon_{11}}-x_1\right)+\left(1-\frac{l^r_{1\varepsilon_{12}}}{l_{1\varepsilon_{12}}}\right)\left(x_{\varepsilon_{12}}-x_1\right)+\left(1-\frac{l^r_{1\varepsilon_{13}}}{l_{1\varepsilon_{13}}}\right)\left(x_{\varepsilon_{13}}-x_1\right)=0$$

that is:

$$\left(1-\frac{l^r_{12}}{l_{12}}\right)\left(x_2-x_1\right)+\left(1-\frac{l^r_{14}}{l_{14}}\right)\left(x_4-x_1\right)+\left(1-\frac{l^r_{16}}{l_{16}}\right)\left(x_6-x_1\right)=0$$

Therefore, referencing the values (10.4), this equation can be also written as:

$$x_2 + x_6 - 2x_1 + \left(1 - \frac{l}{l_{14}}\right)(x_4 - x_1) = 0 \tag{10.6}$$

Likewise it is possible to find the other 11 equations (five more for the *x* coordinates and six for the *y* coordinates). Together these equations are:

$$\begin{cases} x_2 + x_6 - 2x_1 + \left(1 - \frac{l}{l_{14}}\right)(x_4 - x_1) = 0 \\ y_2 + y_6 - 2y_1 + \left(1 - \frac{l}{l_{14}}\right)(y_4 - y_1) = 0 \\ x_1 + x_3 - 2x_2 + \left(1 - \frac{l}{l_{25}}\right)(x_5 - x_2) = 0 \\ y_1 + y_3 - 2y_2 + \left(1 - \frac{l}{l_{25}}\right)(y_5 - y_2) = 0 \\ \dots \end{cases} \tag{10.7}$$

and so on.

At this point, rather than try to resolve this problem directly, let's try to find a short cut. As I've said before, this system of equations, even without taking into account the rigid motions of the solutions, has more than one solution. It's not important for us to find all these solutions, because in this specific problem the different solutions (just two) will all yield the same shape for the model. For our purposes, it is sufficient to find just one solution. Therefore, if we find values for the coordinates that satisfy all the previous equations, we will be sure that we have found one solution to the problem. In order to choose coordinates for this "guess and check" method, we could do some practical reasoning. For example, we could reasonably assume that in the equilibrium configuration all the free masses will be placed as vertices of a regular polygon. Therefore, taking into account the coordinate system indicated in Figure 10.2 and calling *b* the unknown final length of one of the springs inside the hexagon in its equilibrium configuration, we have:

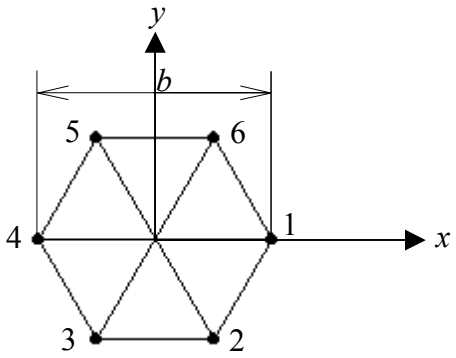

Figure 10.2: attempted solution

$$\begin{cases} x_1 = \frac{1}{2}b & y_1 = 0 \\ x_2 = \frac{1}{4}b & y_2 = -\frac{\sqrt{3}}{4}b \\ x_3 = -\frac{1}{4}b & y_3 = -\frac{\sqrt{3}}{4}b \\ x_4 = -\frac{1}{2}b & y_4 = 0 \\ x_5 = -\frac{1}{4}b & y_5 = \frac{\sqrt{3}}{4}b \\ x_6 = \frac{1}{4}b & y_6 = \frac{\sqrt{3}}{4}b \end{cases} \tag{10.8}$$

If we can find a value for *b* (the final length of the springs inside the hexagon) such that the equations (10.7) are all satisfied, then we'll have a solution to our problem. By substituting the value of the coordinates (10.8) inside the first equation (10.7) we get:

$$\tfrac{1}{4}b + \tfrac{1}{4}b - 2\tfrac{1}{2}b + \left(1 - \frac{l}{b}\right)\left(-\tfrac{1}{2}b - \tfrac{1}{2}b\right) = 0 \tag{10.9}$$

from which follows immediately:

$$b = \tfrac{2}{3}l \tag{10.10}$$

It is easy verify that substituting the coordinate values (10.8) into the equations of the system (10.7) always returns the relation (10.10). This simply means: the solution of our initial problem is provided by the coordinates (10.8) in which the value of *b* is provided by (10.10).

The expression (10.10) also allows us to know the final dimensions of a hexagonal model made with zipsprings on the border and internal springs with a length at rest of *l*. The relation (10.10) can be verified simply by making such a hexagonal model. I'll leave you the pleasure of performing this verification.

10.3 *THE PANDORA'S TOYBOX PHENOMENON*

A few months ago, in the sodarace forum, an interesting debate about an apparently strange model appeared. The model in question was Qwertilliopasd's Pandora's ToyBox. As you can see, it seems that the four springs on the boundary of the box have lost their stiffness. In fact this isn't true. There is an explanation of this phenomenon that is in accordance with the equilibrium equations (10.2), and now we will find it. But before we do, we'll need to discuss the different types of equilibrium.

In how many ways is it possible to bring a sphere to equilibrium on a surface? The immediate answer to this simple question could be: it depends on the surface. Yes, it is true! Let's examine the following three types of surfaces: a concave surface, a convex surface and a horizontal plane surface (see Fig. 10.3).

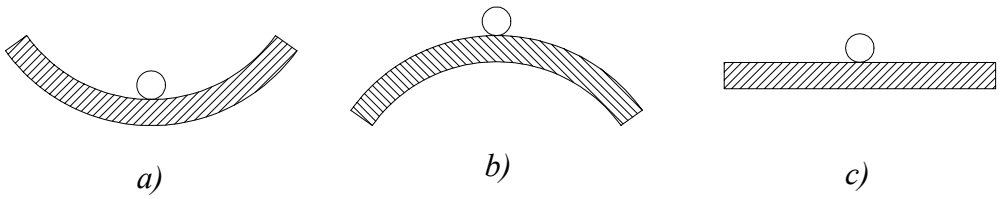

Figure 10.3: kinds of equilibrium

It seems clear that the sphere has a stable equilibrium on the concave surface (Fig. 10.3 *a*). This hypothesis can be tested by moving the sphere a little bit from its equilibrium position and noticing that the sphere immediately returns to its original position. Therefore, we call this case a stable equilibrium. We have another intuitive situation when the sphere is in equilibrium on a convex surface (Fig. 10.3 *b*). This is a case of unstable equilibrium, because if we move the sphere even a tiny amount from its initial equilibrium position, the sphere will definitively lose its equilibrium. Finally, when the sphere is on a horizontal plane (Fig. 10.3 *c*), we have an indifferent equilibrium. In this case the sphere remains in

equilibrium no matter where it is moved on the horizontal plane. The main characteristic of a typical system in indifferent equilibrium is that the equilibrium of the system has total indifference in respect to the configuration of the system.

Returning now to the discussion of Pandora's ToyBox, we can clearly see that it is a case of indifferent equilibrium. In the following paragraphs, I'll show that the equilibrium of the free masses is totally independent of the geometrical configuration of the model.

Pandora's ToyBox is constructed with four zipspring as the contour and two springs of finite and equal length as the diagonals. By examining the behavior of this model it is apparent that the diagonals maintain a constant length, while the springs that make the sides of the rectangle change their lengths apparently without opposition, like springs without stiffness. Immediately, there is one important observation we can make. Because the lengths of the diagonals don't change, the free masses of the model always move in the path of a circle with a diameter equal to the length of the diagonal springs in their compressed state (see Fig. 10.4).

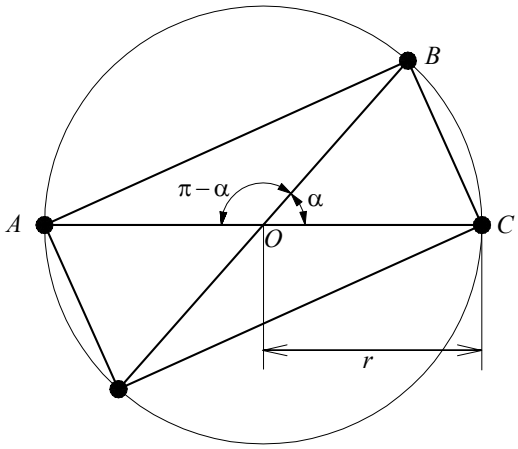

Figure 10.4: The Pandora's ToyBox

Another important observation is that the Pandora's ToyBox, regardless of its specific configuration, always has a rectangular shape. This fact is also confirmed by an important geometrical rule: any triangle inscribed in a half-circumference will be always a rectangular triangle. Indeed, the triangle *ABC* in Figure 10.4 is a rectangular triangle.

As we already know, the tension of a spring depends solely on the extension of the same spring. For this reason, we will now calculate the length of the springs *AB* and *BC* in their

deformed configurations. Thanks to an important Theorem of trigonometry about triangles, looking at the two isosceles triangles *OBC* and *OAB* it is possible to get:

$$\begin{cases} \overline{BC} = \sqrt{2r^2 - 2r^2 \cos\alpha} = \sqrt{2}\, r \sqrt{1-\cos\alpha} \\ \overline{AB} = \sqrt{2r^2 - 2r^2 \cos(\pi-\alpha)} = \sqrt{2}\, r \sqrt{1+\cos\alpha} \end{cases} \tag{10.11}$$

where *r* is the radius of the circumference generated by Pandora's Box (see Fig. 10.4). These lengths coincide with the extension of the relative springs since these springs are zipsprings[31]. The elastic forces acting on the free mass *B* by means of these two zipsprings are:

$$\begin{cases} F_{AB} = k \cdot \overline{AB} = k\sqrt{2}\, r \sqrt{1+\cos\alpha} \\ F_{BC} = k \cdot \overline{BC} = k\sqrt{2}\, r \sqrt{1-\cos\alpha} \end{cases} \tag{10.12}$$

These forces, directed respectively in the directions of the springs *AB* and *BC*, are orthogonal forces. Their resultant *R,* regardless of the structure's configuration, will be always directed in the direction of the spring *BD* (see Fig. 10.5), and its value can be calculated by means of the Theorem of Pythagoras. In the equations (10.12) we see that the values of the forces F_{AB} and F_{BC} are functions of the angle α. However, their resultant *R* can't be, because this resultant compresses the spring *BD* without changing its length. It is exactly this fact that will allow us to verify that the equilibrium of the Pandora's Box is an indifferent equilibrium: the value of the force *R* that compresses the spring *BD* must to be unaffected by the angle α; i.e. the value of the force *R* that compresses the spring *BD* must to be unaffected by the geometrical configuration of the box. Applying the Pythagoras' Theorem, we get:

$$R^2 = F_{AB}^2 + F_{BC}^2 = k^2 2r^2 (1+\cos\alpha) + k^2 2r^2 (1-\cos\alpha) = 4k^2 r^2$$

or:

[31] springs with length at rest equal to zero.

$$R = 2kr \tag{10.13}$$

As you can see, the value of the resultant R is always the same, regardless of the geometrical configuration of the box. This shows without any doubt that the Pandora's ToyBox phenomenon is simply an example of indifferent equilibrium.

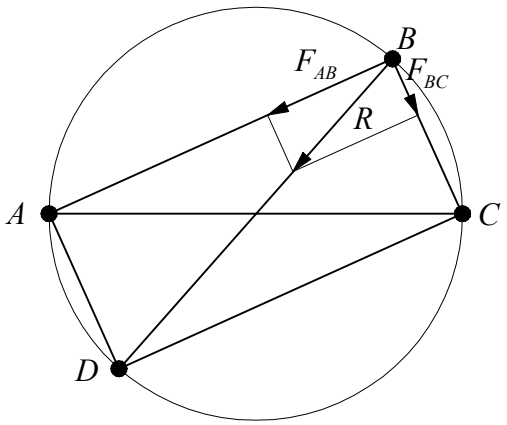

Figure 10.5: Force on the spring BD

10.4 *TENSION SPRINGS / PRE-STRESSED PARTS*

Almost all the models in the zoo contain structures that the sodaconstructor community usually calls "tension springs." These structures consist of a chain of any number of short springs in tension plus another, longer, spring compressed. The importance of these structures is remarkable. It is safe to say that without tension springs a large portion of the models in the zoo would not exist.

Another reason these structures are useful is that there is no standard way to make them. Tension springs are adaptable to a variety of purposes, because it is possible to construct them in a variety of proportions and numbers of internal nodes (see Fig. 10.6).

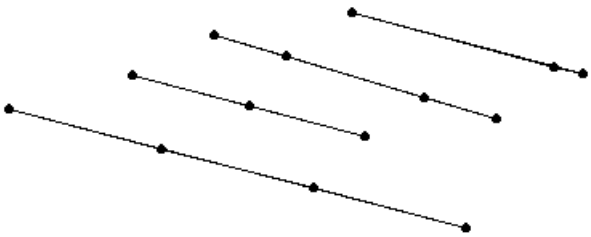

Figure 10.6: Examples of tension springs

Nevertheless, a question about these structures has always remained. What will be the length and configuration of a particular tension spring, based on the initial lengths of the springs (lengths at rest) used? I, myself, have done many attempts in the past in order to get tension springs of particular lengths for my models. After 4 or 5 attempts I was able to get the exact proportion for my tension springs. In this section, I'd like to solve this problem in a more rigorous way.

Before I begin, I'll define the two basic problems to solve regarding tension springs. The first problem is the *direct problem*: How long will the tension spring made with N number of springs of a known length at a rest be? The second problem, instead, is the *inverse problem*: How long should the lengths at rest of the springs be in order to create a tension spring of a specific length? Without any more delay, let us begin the study of the direct problem.

- ***The Direct Problem***

In order to make a tension spring, we need to have N springs in tension plus one in compression. Further, the length at rest of the spring in compression must be greater that the sum of all the lengths at rest of the springs in tension.

We will denote with $l_1^r, l_2^r, \ldots, l_N^r$ the lengths at rest of the N springs in tension inside the tension spring. The length at rest of the compressed spring will be represented by l_0^r. As stated, in order to have a stable tension spring, it will be necessary for:

$$l_0^r > \sum_{i=1}^{N} l_i^r \tag{10.14}$$

Also, we will call $l_0, l_1, l_2, \ldots, l_N$ the final lengths of the previous $N+1$ springs that constitute our tension spring.
By looking at Figure 10.7 it is easy to observe this relationship:

$$l_0 = \sum_{i=1}^{N} l_i \tag{10.15}$$

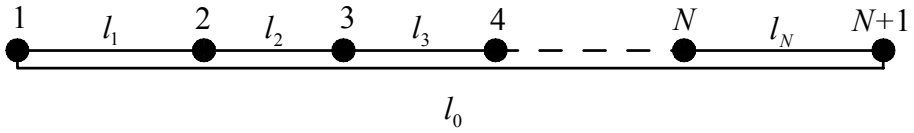

Figure 10.7: Tension spring in its configuration of equilibrium

Finally, we will call Δl_i the stretch of spring number *i*:

$$\Delta l_i = l_i - l_i^r \qquad i = 0,1,2,\ldots,N \tag{10.16}$$

Obviously:

$$\Delta l_1, \Delta l_2, \Delta l_3, \ldots, \Delta l_N > 0; \qquad \Delta l_0 < 0 \tag{10.17}$$

In order to write the equilibrium of all the free masses of the tension spring, it is important to know the values of the forces acting on the nodes. These forces will obviously be related to the stretches Δl_i. In particular, calling F_i the tension on the generic spring *i* (positive for the springs in traction and negative for the spring in compression), applying the well-known Hooke's Law we have:

$$F_i = k \cdot \Delta l_i \qquad i = 0,1,2,\ldots,N \tag{10.18}$$

Figure 10.8 illustrates the forces acting on the free masses of the tension spring.

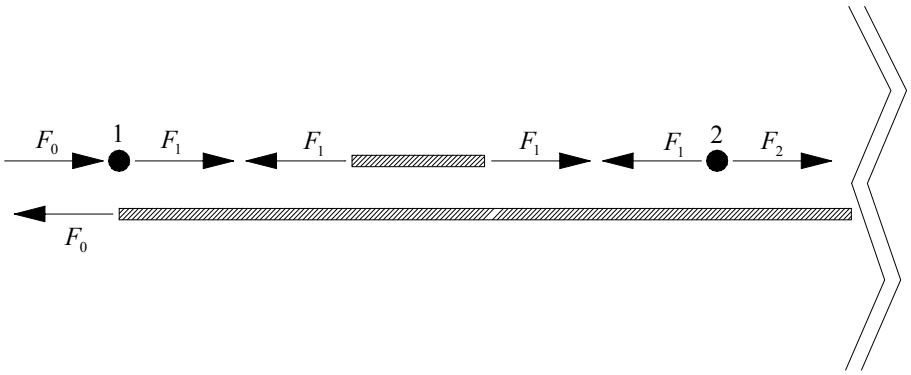

Figure 10.8: Forces acting on the free masses of the tension spring.

Looking at Figure 10.8 it is possible to understand how, based on the static equilibrium of the free masses, the following relations are valid:

$$\begin{cases} -F_0 = F_1 \\ F_1 = F_2 \\ F_2 = F_3 \\ \vdots \\ F_{N-1} = F_N \\ F_N = -F_0 \end{cases}$$

that is:

$$-F_0 = F_1 = F_2 = F_3 = \cdots = F_{N-1} = F_N \tag{10.19}$$

With the combination of (10.18) and (10.19) immediately follows this important relation:

$$-\Delta l_0 = \Delta l_1 = \Delta l_2 = \Delta l_3 = \cdots = \Delta l_{N-1} = \Delta l_N = \Delta l \tag{10.20}$$

That is, all the springs are subject to the same tension, in absolute value. Specifically, the springs $1, 2, \ldots, N$ are subjected to the unknown extension Δl, while the spring 0 is subjected to the contraction $-\Delta l$.

Thanks to (10.20), from (10.16) follows:

$$\begin{cases} l_0 = l_0^r - \Delta l \\ l_i = l_i^r + \Delta l \quad i = 1, 2, \ldots, N \end{cases} \tag{10.21}$$

Substituting these relations into the equation (10.15) we get:

$$l_0^r - \Delta l = \sum_{i=1}^{N} \left(l_i^r + \Delta l \right) = N\Delta l + \sum_{i=1}^{N} l_i^r$$

or better:

$$\Delta l = \frac{l_0^r - \sum_{i=1}^{N} l_i^r}{N+1} \tag{10.22}$$

This expression is very important, since gives us the value of the stretch Δl of each spring in the tension spring. From this positive value[32], thanks to (10.21) it is possible to get the final lengths of all the springs. In particular, the final length of the tension spring will be:

$$l_0 = l_0^r - \frac{l_0^r - \sum_{i=1}^{N} l_i^r}{N+1} = \frac{N\, l_0^r + \sum_{i=1}^{N} l_i^r}{N+1} \tag{10.23}$$

It is very simple to verify this relation by constructing a tension spring and measuring all the lengths at rest, as well as the final length. I will leave this exercise to you.

- ***The Inverse Problem***

In the inverse problem, we know the final lengths $l_0, l_1, l_2, \ldots, l_N$ of the springs, and we need to find lengths at rest of all the springs in our tension spring.

[32] We can safely say this because of (10.14).

Taking into account what we have already obtained in the previous direct problem, the solution of the inverse problem is fairly trivial. We can simply follow these three steps:

1. Choose an arbitrary value for the length at rest l_0^r.
2. Calculate the stretch value $\Delta l = l_0^r - l_0$.
3. To calculate the length at rest of all the springs by means of the formula $l_i^r = l_i - \Delta l \quad i = 1, 2, \ldots, N$

And that's it. Once again, I will leave you with the pleasure of testing this procedure.

10.5 *LINEAR MOTORS*

For the last application of the quasi-static procedure, I just can't help but examine my beloved linear motors. These structures have given me a lot of satisfaction, so I feel obligated to return the favor. In this last section of the last chapter of this long paper about soda physics, I will talk about the physical-mathematical behavior of these amazing structures.

The mathematical characterization of these structures requires the study of all the possible configurations that a linear motor can assume in its movement, so here I will limit myself to just an explanation of the 3 point linear motor. Nevertheless, the procedure that I will adopt for this study can also be applied to more complex linear motors.

To begin, it is important to explain how it is possible to make a 3pt linear motor. It is very simple: a 3pt linear motor is a triangle with 3 equal muscles timed respectively at the 1/3 divisions of the wave, and simulated with maximum wave and muscle amplitudes. You can view a 3pt linear motor here.

When the simulation is started, the triangular model will assume a typical linear configuration and the masses will start to move back and forth with periodical cadence.

Let us start the study of this motor by taking into account the placement of the three free masses indicated in the following Figure 10.9:

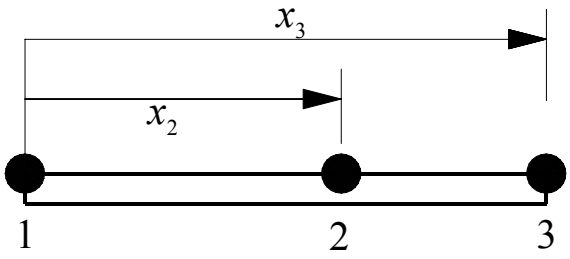

Figure 10.9: 3pt linear motor

As indicated, we will report the position of the masses with respect to mass 1. In this way, we will eliminate from the resolution of the problem all the unessential rigid motions.
Taking into account what we know about the muscles of this motor, we can say that their lengths at rest at a generic instant of time *t* will be:

$$\begin{cases} l_{12}^r(t) = l\left[1+\sin\omega t\right] \\ l_{23}^r(t) = l\left[1+\sin\left(\omega t+\frac{2}{3}\pi\right)\right] \\ l_{13}^r(t) = l\left[1+\sin\left(\omega t+\frac{4}{3}\pi\right)\right] \end{cases} \tag{10.24}$$

while the effective lengths of the same three muscles at a generic instant of time *t* will be (see Fig. 10.9):

$$\begin{cases} l_{12} = x_2 \\ l_{13} = x_3 \\ l_{23} = x_3 - x_2 \end{cases} \tag{10.25}$$

Therefore, from (10.24) and (10.25) follows the stretch of the three springs:

$$\begin{cases} \Delta l_{12} = l_{12} - l_{12}^r = x_2 - l_{12}^r \\ \Delta l_{13} = l_{13} - l_{13}^r = x_3 - l_{13}^r \\ \Delta l_{23} = l_{23} - l_{23}^r = x_3 - x_2 - l_{12}^r \end{cases} \tag{10.26}$$

Like in the previous problem, when the value of the stretch of the muscle $\Delta l > 0$, the muscle is in extension, while when the stretch of the muscle $\Delta l < 0$, the muscle is in compression. Applying Hooke's Law, we get also the values of the tensions of the muscles:

$$\begin{cases} F_{12} = k\,\Delta l_{12} \\ F_{13} = k\,\Delta l_{13} \\ F_{23} = k\,\Delta l_{23} \end{cases} \tag{10.27}$$

Likewise, the previous case from the equilibrium of the free masses 2 and 3 follows:

$$\begin{cases} F_{12} - F_{23} = 0 \\ F_{13} + F_{23} = 0 \end{cases} \tag{10.28}$$

Thanks to the (10.27), this can be written as:

$$\begin{cases} \Delta l_{12} - \Delta l_{23} = 0 \\ \Delta l_{13} + \Delta l_{23} = 0 \end{cases} \tag{10.29}$$

Therefore, taking into account (10.26), finally we get:

$$\begin{cases} 2x_2 - x_3 = l_{12}^r - l_{23}^r \\ 2x_3 - x_2 = l_{13}^r + l_{23}^r \end{cases} \tag{10.30}$$

This is a linear system of two equations in the two unknown coordinates x_2 and x_3. The solution of this system of equation is:

$$\begin{cases} x_2 = \frac{1}{3}\left(2l_{12}^r + l_{13}^r - l_{23}^r\right) \\ x_3 = \frac{1}{3}\left(l_{12}^r + 2l_{13}^r + l_{23}^r\right) \end{cases} \tag{10.31}$$

At this point, by simply substituting the expression of the lengths at rest (10.24) in the solution (10.31), we will get part of the solution: the laws of variation in time of the coordinates $x_2(t)$ and $x_3(t)$.

This solution, unfortunately, isn't enough to fully describe the behavior of the 3pt linear motor, because the relations (10.25) are only valid when the following conditions are satisfied:

$$\begin{cases} l_{12} = x_2 > 0 \\ l_{13} = x_3 > 0 \\ l_{23} = x_3 - x_2 > 0 \end{cases} \tag{10.32}$$

During the movement of the 3pt linear motor, the free masses continually change their respective positions, so that the conditions (10.32) aren't always satisfied. Therefore, the solution provided by (10.24) and (10.31) is only valid when the configuration is like that reported in Figure 10.9.

In order to get a general solution for the 3pt linear motor, it is indispensable to study all the possible reciprocal positions of the three free masses during their movement. I've already done this complete study but I will avoid repeating this work here. It will suffice to say that there are a total of six possible configurations of the three free masses during their motion. Therefore, in order to completely describe the behavior of the 3pt linear motor, we'll need six different solutions like (10.31).

In the diagram of Figure 10.10, I've reported all six different solutions. The behavior of the coordinates $x_2(t)$ and $x_3(t)$ is displayed in green and red curves. It is nice to note that these two curves were been obtained by taking different pieces of the six solutions.

Another thing to note is that the length of period for the motion of the free masses is double the period of the muscles.

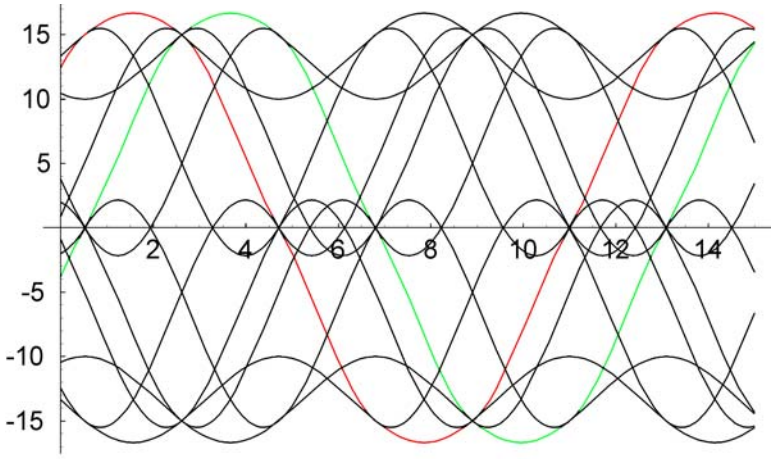

Figure 10.10: Displacement of the masses in a 3pt linear motor.

The diagram of Figure 10.10 was been obtained by taking an initial length of the muscles equal to $l = 10\ pxl$ and a pulsation equal to $\omega = 1\ rad/frm$. The horizontal axis represents the time expressed in frames, while the vertical axis represents the values of the displacement $x_2(t)$ and $x_3(t)$ expressed in pixels.

11. CONCLUSIONS AND ACKNOWLEDGEMENTS

The first and only thing that I wish to say in conclusion about this work is that, in the full spirit of the sodaplay web site, I've done all this just for fun.

I feel obligated to give a special thanks to my friend Lectvay for his enthusiastic and patient support of this work. Indeed, without his contribution of the revision of my text, this job would be particularly incomprehensible. (Editor's note: Jeckyll is too modest about his very respectable command of English as a second language.)

Finally a special thanks goes to Ed Burton, the man who made this beautiful "piano".

APPENDIX A: THE PENDULUM

The pendulum was the first instrument used in this paper in order to find the value of the gravity constant *g* with accuracy. As you can see in Figure A.1, its structure is very simple. It consists of an inextensible wire of negligible mass connected to a mass at a fixed point. The system is subjected to a uniform gravitation field in the negative direction of the *y*-axis.

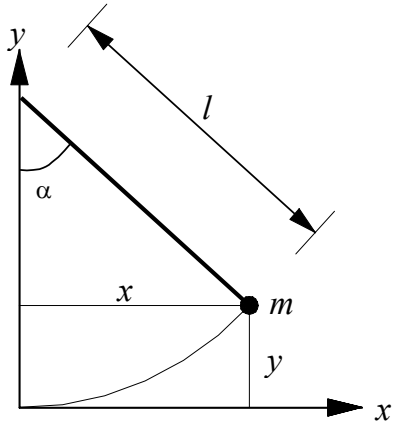

Figure A.1: System of reference for the pendulum.

To get the differential equation of motion for the pendulum, we will apply the conservation of energy.
The coordinates of the pendulum mass are:

$$\begin{cases} x = l\sin\alpha \\ y = l(1-\cos\alpha) \end{cases} \tag{A.1}$$

therefore, their time derivatives are:

$$\begin{cases} \dot{x} = l\ \dot{\alpha}\cos\alpha \\ \dot{y} = l\ \dot{\alpha}\sin\alpha \end{cases} \tag{A.2}$$

To find the kinetic energy we need the square of the velocity. Therefore, from (A.2) follows:

$$v^2 = l^2\,\dot{\alpha}^2 \tag{A.3}$$

The total energy E of a system is given by the sum of the kinetic energy and the potential energy. Therefore, we have:

$$E = \tfrac{1}{2} m v^2 + m\,g\,y = \tfrac{1}{2} m l^2 \dot{\alpha}^2 + mgl(1-\cos\alpha) \tag{A.4}$$

Since the total energy is constant in time (conservation of energy):

$$\frac{d\,E}{d\,t} = 0 \tag{A.5}$$

Taking into account the expression (A.4) of the energy, from (A.5) follows:

$$\frac{d\,E}{d\,t} = \tfrac{1}{2} m l^2\, 2\dot{\alpha}\ddot{\alpha} + mgl\dot{\alpha}\sin\alpha = 0 \tag{A.6}$$

that is:

$$\ddot{\alpha} + \omega^2 \sin\alpha = 0 \tag{A.7}$$

where we have assumed the position:

$$\omega^2 = \frac{g}{l} \tag{A.8}$$

The equation (A.7) represents the differential equation of motion for the pendulum. This equation isn't linear, so its solution isn't so simple to get. In order to get a simple solution for the above differential equation it will be necessary to assume that the maximum amplitude of the pendulum oscillation is small enough that it be possible to approximate:

$$\sin\alpha \approx \alpha \tag{A.9}$$

In the virtual experiments performed to find the constant *g* of the applet, a maximum value of α equal to $\alpha_{max} = 0.05\ rad$ has been assumed. This value certainly guarantees for the relation (A.9) errors less than the experimental errors in the above virtual experiments.
Thanks to the relation (A.9), the differential equation (A.7) becomes:

$$\ddot{\alpha} + \omega^2 \alpha = 0 \tag{A.10}$$

As will be shown in detail in Appendix B, the solution of this linear differential equation is:

$$\alpha(t) = \rho\cos(\omega t - \varphi) \tag{A.11}$$

where:

$$\begin{cases} \rho = \sqrt{\alpha_0^2 + \left(\dfrac{\dot{\alpha}_0}{\omega}\right)^2} \\ \tan\varphi = \dfrac{\dot{\alpha}_0}{\alpha_0\ \omega} \end{cases} \tag{A.12}$$

and:

$$\begin{cases} \alpha_0 = \alpha(0) \\ \dot{\alpha}_0 = \dot{\alpha}(0) \end{cases} \tag{A.13}$$

From the solution (A.11) it is possible to get the value of the period *T* of a complete oscillation of the pendulum; i.e. the minimum value that satisfies the relation:

$$\cos(\omega t - \varphi) = \cos(\omega(t+T) - \varphi) \qquad \forall t \tag{A.14}$$

This value is:

$$T = \frac{2\pi}{\omega} \tag{A.15}$$

or better:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{A.16}$$

From this relation follows immediately:

$$g = \frac{4\pi^2 l}{T^2} \tag{A.17}$$

This relation is very important, because it allows us to find the value of the gravity acceleration g by simply measuring the period T of a complete oscillation of the pendulum.

APPENDIX B: DAMPED FREE VIBRATIONS (SINGLE DEGREE OF FREEDOM)

Let us consider a system with just one degree of freedom like that schematized in Figure B.1.

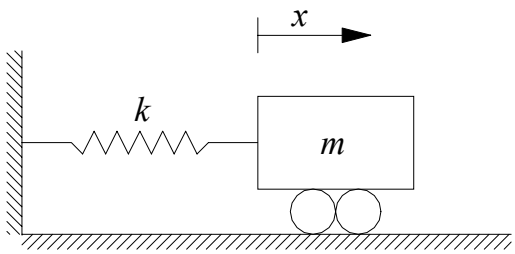

Figure B.1: System with one degree of freedom

This system consists of a mass that is free to move on the surface of a plane. This mass is connected to a rigid wall by means of a spring of stiffness *k*. In its movement, this mass will also be subjected to the air friction *f*. In order to describe mathematically the movement of this mass, the origin will be placed on the left side of the mass in its configuration of static equilibrium. The positive direction of the displacements, velocities, acceleration and forces will be from the left to the right.

The forces acting on the mass in its movement are the elastic force F_e and the damping force F_v. Their values are:

$$\begin{cases} F_e = -k\,x \\ F_v = -f\,\dot{x} \end{cases} \tag{B.1}$$

in which the dependence from time is understood. From Newton's second law follows:

$$F = F_e + F_v = m\,\ddot{x} \tag{B.2}$$

that is:

$$m\,\ddot{x} + f\,\dot{x} + k\,x = 0 \tag{B.3}$$

or better:

$$\ddot{x} + \frac{f}{m}\dot{x} + \frac{k}{m}x = 0 \tag{B.4}$$

In order to simplify the mathematical treatment of this problem, we will assume the following positions:

$$\begin{cases} \dfrac{k}{m} = \omega^2 \\ \dfrac{f}{m} = 2\omega\xi \end{cases} \tag{B.5}$$

so that the differential equation (B.4) can be written as:

$$\ddot{x} + 2\omega\xi\,\dot{x} + \omega^2\,x = 0 \tag{B.6}$$

This is a linear homogeneous differential equation with constant coefficients. Its solution provides the law of the displacement of *x* over time.
We are looking for solutions to the equations (B.6) in the form:

$$x(t) = e^{\lambda t} = \exp\{\lambda\, t\} \tag{B.7}$$

The first and second derivatives of this function are:

$$\begin{cases} \dot{x}(t) = \lambda \exp\{\lambda\, t\} \\ \ddot{x}(t) = \lambda^2 \exp\{\lambda\, t\} \end{cases} \tag{B.8}$$

Therefore, substituting the relations (B.7) and (B.8) in the differential equation (B.6) we get the equation:

$$\lambda^2 \exp\{\lambda t\} + 2\omega\xi\lambda \exp\{\lambda t\} + \omega^2 \exp\{\lambda t\} = 0 \tag{B.9}$$

which is a 2nd degree algebraic equation:

$$\lambda^2 + 2\omega\xi\lambda + \omega^2 = 0 \tag{B.10}$$

The two solutions of this equation are:

$$\lambda_{1,2} = \omega\left(-\xi \pm \sqrt{\xi^2 - 1}\right) \tag{B.11}$$

At this point, before acquiring the solutions of the original differential equation, we must distinguish between the following three cases:

1. $\xi = 1$
2. $\xi > 1$
3. $\xi < 1$

In the firsts two cases, the values (B.11) of the constants λ will both be real and negative numbers. In these two cases, the solutions of the differential problem won't include any oscillatory movement. The mass will tend to return directly to its quiet configuration. The limit value of the coefficient ξ that distinguishes this particular behavior of the mass from the typical oscillatory movement is $\xi = 1$. Taking into account the positions (B.5), we have $\xi = 1$ when:

$$f = 2\sqrt{k\,m} \tag{B.12}$$

This is the value of the critical damping. Whenever the damping coefficient f is greater than $2\sqrt{k\,m}$, the oscillatory behavior of the mass will vanish. These are cases of *overdamping*.

Taking into account the position assumed for the stiffness *k* and the damping coefficient *f,* the relation (B.12) can also be written as:

$$\bar{f} = 2\sqrt{\bar{k}}$$

Whenever we set the damping value $\bar{f}$ greater than $2\sqrt{\bar{k}}$ our models will not present oscillatory behavior. I usually use overdamping when making complex models, where I need to move free masses that are already connected out of the way in order to make other connections. Then, I simply start the simulation of the model, and the dislocated free masses slowly return to their places, without causing oscillations in the model.
We will not study the overdamping case, because it has no application in this paper. Therefore, I'll restrict myself to write just the final solution for the critical damping (B.12). When $\xi = 1$ we will have:

$$x(t) = \left[x_0(1+\omega t) + v_0\, t\right]\exp\{-\omega t\} \tag{B.13}$$

where:

$$\begin{cases} x_0 = x(0) \\ v_0 = \dot{x}(0) \end{cases} \tag{B.14}$$

The typical behavior of a system with critical damping is reported in the diagram of Figure B.2.

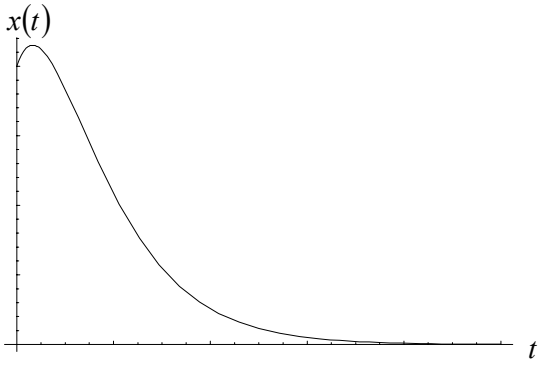

Figure B.2: Displacement of a system with critical damping.

In order to define a procedure for the determination of the damping parameter *f* it is important to study the third case, where $\xi < 1$. In this case, the values of the constants λ in (B.11) can be written as:

$$\lambda_{1,2} = \omega\left(-\xi \pm i\sqrt{1-\xi^2}\right) \tag{B.15}$$

in which has been introduced the imaginary unit $i = \sqrt{-1}$. The values of the previous constant (B.15) can be also written as:

$$\lambda_{1,2} = -\xi\,\omega \pm i\,\omega_D \tag{B.16}$$

in which has been introduced the *damped frequency*:

$$\omega_D = \omega\sqrt{1-\xi^2} \tag{B.17}$$

Substituting the complex values of the constant λ in (B.7), we get the following two independent solutions for the differential equation (B.6):

$$\begin{cases} x_1(t) = \exp\{-\xi\,\omega\,t\}\exp\{i\,\omega_D\,t\} \\ x_2(t) = \exp\{-\xi\,\omega\,t\}\exp\{-i\,\omega_D\,t\} \end{cases} \tag{B.18}$$

These two solutions, thanks to the well-known Euler formula:

$$\exp\{i\,\alpha\} = \cos\alpha + i\sin a \tag{B.19}$$

can be also written as:

$$\begin{cases} x_1(t) = \exp\{-\xi\,\omega t\}\left[\cos(i\,\omega_D\,t) + i\sin(i\,\omega_D\,t)\right] \\ x_2(t) = \exp\{-\xi\,\omega t\}\left[\cos(i\,\omega_D\,t) - i\sin(i\,\omega_D\,t)\right] \end{cases} \tag{B.20}$$

One of the primary properties of a linear differential equation is that if its solution is combined in a linear way, the result of this combination is itself a solution of the original differential equation. Therefore, rather than use the complex solutions (B.20), it will be more convenient to use the following two different solutions derived from (B.20) by means of two special linear combinations:

$$\begin{cases} \bar{x}_1(t) = \dfrac{x_1(t) + x_2(t)}{2} = \exp\{-\xi\,\omega t\}\cos(\omega_D\,t) \\ \bar{x}_2(t) = \dfrac{x_1(t) - x_2(t)}{2\,i} = \exp\{-\xi\,\omega t\}\sin(\omega_D\,t) \end{cases} \tag{B.21}$$

Therefore, the general solution for the linear differential equations (B.6) can be written as:

$$x(t) = \exp\{-\xi\,\omega t\}\left[A_1\cos(\omega_D\,t) + A_2\sin(\omega_D\,t)\right] \tag{B.22}$$

The values of the constants A_1 and A_2 can be written in terms of the following initial conditions:

$$\begin{cases} x_0 = x(0) \\ v_0 = \dot{x}(0) \end{cases} \tag{B.23}$$

After slight mathematical manipulation, we find the following values of the constants A_1 and A_2 in terms of the initial conditions:

$$\begin{cases} A_1 = x_0 \\ A_2 = \dfrac{v_0 + \xi\,\omega\,x_0}{\omega_D} \end{cases} \tag{B.24}$$

so that the general solution (B.22) can be also written:

$$x(t) = \exp\{-\xi\,\omega\,t\}\left[x_0\,\cos(\omega_D\,t) + \frac{v_0 + \xi\,\omega\,x_0}{\omega_D}\sin(\omega_D\,t)\right] \tag{B.25}$$

Another and more compact expression for the solution (B.25) can be obtained by simply imposing the following two positions:

$$\rho\sin\varphi = \frac{v_0 + \xi\,\omega\,x_0}{\omega_D};\quad \rho\cos\varphi = x_0 \tag{B.26}$$

Indeed, substituting in (B.25) the positions (B.26) we get the following expression for the solution of the linear differential equation (B.6):

$$x(t) = \rho\exp\{-\xi\,\omega\,t\}\cos(\omega_D t - \varphi) \tag{B.27}$$

where from (B.26) follows immediately:

$$\begin{cases} \rho = \sqrt{x_0^2 + \left(\dfrac{v_0 + \xi\,\omega\,x_0}{\omega_D}\right)^2} \\ \tan\varphi = \dfrac{v_0 + \xi\,\omega\,x_0}{x_0\,\omega_D} \end{cases} \tag{B.28}$$

The solution (B.27) is particularly eloquent, because it allows us to understand the behavior of a damped free oscillation. Indeed, this solution is essentially composed of two parts: the amplitude of the oscillation that decreases with law $\rho \exp\{-\xi \omega t\}$, and the oscillatory part $\cos(\omega_D t - \varphi)$ with period equal to $T = 2\pi/\omega_D$. In the following Figure B.3 it is possible to see the typical behavior of a single degree of freedom damped free oscillation.

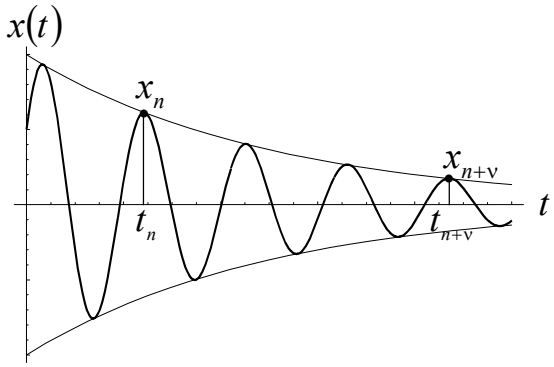

Figure B.3: Behavior of a damped free oscillation.

The solution (B.27) of the differential equations (B.6) is very important, because it will allow us to find a relation for the experimental determination of the damping parameter *f*. By simply measuring the amplitude of two different peaks of the oscillation, the value of the damping parameter *f* can be determined.

Obviously, we will have peaks of oscillations approximately when:

$$\cos(\omega_D t - \varphi) = 1 \tag{B.29}$$

This happens with periodical cadence established by the period $T = 2\pi/\omega_D$. Therefore, by measuring the peak x_n that occurs at time t_n such that $\cos(\omega_D t_n - \varphi) = 1$, and then, after $\nu \geq 1$ complete oscillation, measuring the peak $x_{n+\nu}$ that occurs at time $t_{n+\nu} = t_n + \nu T$, we find:

$$\begin{cases} x_n = \rho \exp\{-\xi \omega t_n\} \\ x_{n+\nu} = \rho \exp\{-\xi \omega t_{n+\nu}\} = \rho \exp\{-\xi \omega (t_n + \nu T)\} = \rho \exp\{-\xi \omega t_n\} \exp\{-\xi \omega \nu T\} \end{cases} \tag{B.30}$$

in which the known quantities are the already measured values of x_n and $x_{n+\nu}$. From the ratio of these two quantities follows:

$$\frac{x_n}{x_{n+\nu}} = \exp\{\xi\,\omega\,\nu\,T\} = \exp\left\{\xi\,\omega\,\nu\,\frac{2\pi}{\omega_D}\right\} \tag{B.31}$$

Taking the natural logarithm of both sides of (B.31) we have:

$$\ln\left(\frac{x_n}{x_{n+\nu}}\right) = \xi\,\omega\,\nu\,\frac{2\pi}{\omega_D} \tag{B.32}$$

or better:

$$\delta = \xi\,\omega\,\nu\,\frac{2\pi}{\omega_D} \tag{B.33}$$

in which has been introduced the logarithmic parameter:

$$\delta = \ln\left(\frac{x_n}{x_{n+\nu}}\right) \tag{B.34}$$

Remembering the previous positions (B.5) and (B.17), from (B.33) follows:

$$\delta^2 = \frac{4\nu^2\pi^2 f^2}{4\,k\,m - f^2} \tag{B.35}$$

or better:

$$f = \frac{2\sqrt{k\,m}}{\sqrt{1+\left(\frac{2\nu\pi}{\delta}\right)^2}} \tag{B.36}$$

Thanks to (B.34) and (B.36), it is possible to experimentally determine the value of the damping constant *f*.

• ***The Undamped Free Vibrations***

In the simplest case, where the mass of Figure B.1 isn't subjected to any air friction, $f = 0$ and, consequently, $\xi = 0$ so that the differential equation of motion (B.6) becomes:

$$\ddot{x} + \omega^2\, x = 0 \tag{B.37}$$

The general solution for this differential equation can be obtained from the solution (B.27) by simply taking $\xi = 0$. Therefore we have:

$$x(t) = \rho\cos(\omega t - \varphi) \tag{B.38}$$

where the constants ρ and φ are related to the initial position x_0 and velocity v_0 by means of the relations:

$$\begin{cases} \rho = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \\ \tan\varphi = \frac{v_0}{x_0\,\omega} \end{cases} \tag{B.39}$$

From the solution (B.38) it is possible to see that the oscillations of this system won't have any diminution of their max amplitude over time. This means that in this kind of oscillation, there isn't any loss of energy over time. In order to create this phenomenon using the sodaconstructor applet, simply make friction *f* equal to zero.

From the solution (B.38) it is also possible to see how the period T of a complete oscillation is:

$$T = \frac{2\pi}{\omega} \tag{B.40}$$

From this relation, taking into account the original expression of the frequency ω reported in (B.5), follows:

$$\omega^2 = \frac{k}{m} = \frac{4\pi^2}{T^2} \tag{B.41}$$

or better:

$$k = \frac{4\pi^2}{T^2} m \tag{B.42}$$

Thanks to this relation, by simply measuring the period of a complete oscillation of an undamped system like that indicated in Figure B.1, it is possible to get the value of the stiffness of a spring.

APPENDIX C: OTHERS NUMERICAL TECHNIQUES

In chapter 9, I presented the simplest numerical technique that can be used to solve a set of differential equations: the finite difference method. Nevertheless, this technique isn't particularly efficient[33] and can't be applied in many cases of physical relevance. Moreover, as can be discovered by reading an old discussion in the sodarace forum, the numerical algorithm used in the sodaconstructor applet is the Euler method. For these reasons, I've decided to explain Euler and other numerical algorithms in this appendix.

- ***The Euler Method: Non-Linear Differential Equation of the First Order***

The most general expression for a non-linear differential equation of first order is:

$$f\left[y'(x), y(x), x\right] = 0 \tag{C.1}$$

where f is a generic function of three variables[34], while $y(x)$ is the unknown function. Generally speaking it is possible to say that this differential equation has infinite solutions; i.e. there are infinite functions $y(x)$ that satisfy the equation (C.1). Nevertheless, here we are concerned with numerical techniques that allow us to find one particular solution by means of its numerical values. Therefore, we need to specify which of the infinite solutions of (C.1) we are looking for. At this point the question is: how is it possible to specify what solution we are looking for? The answer is simple: choosing a starting point from which the numerical technique will be able to "construct" the entire solution. This starting point is normally specified by means of the *boundary condition* that, for the differential equation (C.1), takes the expression:

$$y(x_0) = y_0 \tag{C.2}$$

where x_0 and y_0 are the coordinates of a particular point on the *xy* plane.

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[33] I'll clarify better what I exactly mean when I speak about efficiency at the end of this appendix

[34] The expression of this function is well known since it depends of the particular problem that we are studying.

It is possible to see that under certain conditions there is just one solution of the differential equation (C.1) that will pass through the point x_0, y_0.

In order to apply Euler's method, it is important to rewrite the differential equation (C.1) in its normal form:

$$y'(x) = f[y(x), x] \tag{C.3}$$

where *f* is now a different function (of two variables). Obviously we also need the boundary condition (C.2).

Since the first derivative of a function represents the slope of that function at a particular point, we are immediately able to recognize that the differential equations (C.3) together with its boundary condition (C.2) give us the slope "*s*" of the unknown function $y(x)$ at the point x_0, y_0 (see Figure C.1):

$$s_0 = y'(x_0) = f[y(x_0), x_0] = f(y_0, x_0) \tag{C.4}$$

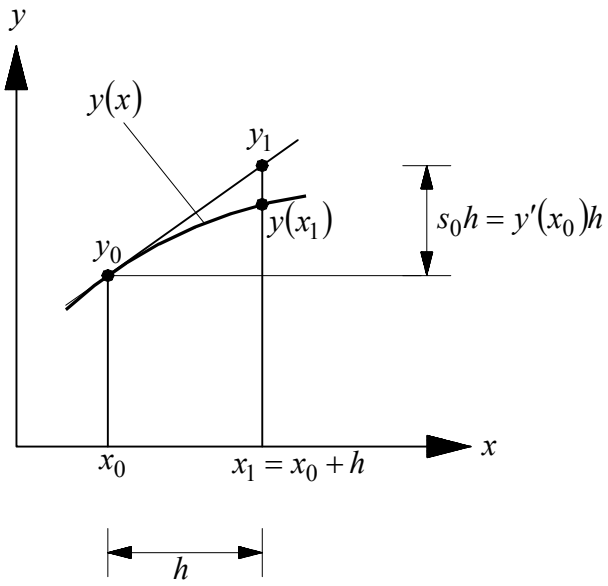

Figure C.1: The geometric meaning of the first derivative

Therefore, if we choose a finite interval *h*, we'll be able to get a second point on our unknown function taking:

$$\begin{cases} x_1 = x_0 + h \\ y_1 = y_0 + s_0 h \end{cases} \tag{C.5}$$

However, looking at Figure C.1, it is clear that the point x_1, y_1 isn't really a point on our unknown function $y(x)$. The point x_1, y_1 is just an approximation of the true (but unknown) point $x_1, y(x_1)$. Obviously, this approximation is due to the substitution of the bent piece of the function $y(x)$ in the interval $(x_0, x_0 + h)$ with a straight line having the same slope as $y(x)$ at x_0.

For the sake of clearness, a big interval was chosen for the interval *h* in Figure C.1. But, as you can see in Figure C.1, the error $y_1 - y(x_1)$ gets smaller as *h* decreases, so that the point x_1, y_1 obtained using (C.4) and (C.5) will be a good approximation of a point of $y(x)$ if we have a small value for *h*.

Now, having a second point of our function, using the same procedure we will be able to find a third point using the new formulas:

$$\begin{cases} s_1 = f(y_1, x_1) \\ x_2 = x_1 + h = x_0 + 2h \\ y_2 = y_1 + s_1 h \end{cases} \tag{C.6}$$

and so on for more and more points of $y(x)$. At step number *i* we will have:

$$\begin{cases} s_{i-1} = f(y_{i-1}, x_{i-1}) \\ x_i = x_0 + i \cdot h \\ y_i = y_{i-1} + s_{i-1} h \end{cases} \tag{C.7}$$

These relations will allow us to find all the points of our unknown function $y(x)$, step by step.

• ***The Euler Method: System of Non-Linear Differential Equations of the First Order***

In the previous section I talked about the use of Euler's method when our mathematical problem is represented by a non-linear differential equation of the first order. Here I'll talk about the use of the same numerical algorithm when our problem is represented by a system of non-linear differential equations of the first order.

Since it is trivial to get the generalization to a system of *N* differential equations, here I'll restrict myself to talk about a simple system of two differential equations of first order in two unknown functions. This system in its normal form is:

$$\begin{cases} y'(x) = f[y(x), z(x), x] \\ z'(x) = g[y(x), z(x), x] \end{cases} \tag{C.8}$$

where $y(x)$ and $z(x)$ are the unknown functions while *f* and *g* are two generic functions with three variables[35]. Obvioulsy, along with this system of differential equations we must also specify the correspondent boundary conditions. These, for our problem, are:

$$\begin{cases} y(x_0) = y_0 \\ z(x_0) = z_0 \end{cases} \tag{C.9}$$

Like in the previous case, taking into account the geometrical meaning of the first derivative of a function, from (C.8) and (C.9) we get the slopes s_0^y and s_0^z of the functions $y(x)$ and $z(x)$ respectively in correspondence with the abscissa x_0:

$$\begin{cases} s_0^y = f(y_0, z_0, x) \\ s_0^z = g(y_0, z_0, x) \end{cases} \tag{C.10}$$

[35] As in the previous case, these functions are well known since they depend on the particular problem that we are studying.

Therefore, fixing a small finite interval *h*, the approximate values of the functions $y(x)$ and $z(x)$ in correspondence with the abscissa $x_0 + h$ will be:

$$\begin{cases} y_1 = y_0 + s_0^y h \\ z_1 = z_0 + s_0^z h \end{cases} \tag{C.11}$$

Thanks to these new values we will be able to iterate the same procedure in order to get a sequence of points x_i, y_i, z_i that represents our approximate solution to the problem defined by the system of differential equations (C.8) and the relative boundary conditions (C.9).
The generic expressions for step number *i* of the iteration are:

$$\begin{cases} x_i = x_0 + i \cdot h \\ s_{i-1}^y = f(y_{i-1}, z_{i-1}, x_{i-1}) \\ s_{i-1}^z = g(y_{i-1}, z_{i-1}, x_{i-1}) \\ y_i = y_{i-1} + s_{i-1}^y h \\ z_i = z_{i-1} + s_{i-1}^z h \end{cases} \tag{C.12}$$

• ***The Euler Method: Non-Linear Differential Equations of the Second Order***

Usually, the equations of motion for a physical system are expressed in terms of differential equations of the second order. For this reason we will now discuss the application of the Euler method in the resolution of a non-linear differential equation of the second order. This equation in its normal form has the expression:

$$y''(x) = f[y'(x), y(x), x] \tag{C.13}$$

where $y(x)$ is the unknown function and *f* is a particular function with three variables.
Once again, you can see that in order to get a specific solution of this differential equation (C.13) we need an appropriate number of boundary conditions. In this case we have:

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases} \tag{C.14}$$

with the constants x_0, y_0, y'_0.

As we know, Euler's method is based on the geometric meaning of the first derivative of a function, so this method seemingly can't be applied to the differential equation (C.13), where the second derivative of the function $y(x)$ appears. However, I'll show you how it is always possible to transform a differential equation of the second order (C.13) into a system of two differential equations of the first order. To do that, let's examine the following position:

$$z(x) = y'(x) \tag{C.15}$$

Indeed, using this relation, the differential problem defined by the relations (C.13) and (C.14) becomes:

$$\begin{cases} y'(x) = z(x) \\ z'(x) = f[z(x), y(x), x] \end{cases} \tag{C.16}$$

together with the following boundary conditions:

$$\begin{cases} y(x_0) = y_0 \\ z(x_0) = y'_0 \end{cases} \tag{C.17}$$

It's easy to recognize that the differential problem shown in equations (C.16) and (C.17) is an instance of the problem that wediscussed in the previous section. Therefore, the Euler method is applicable.

• ***The Euler Method: Systems of Differential Equations of the Second Order***

The equations of the motion of a generic model were formulated in chapter 8 of this paper. Looking at (8.10) it is easy to see that these equations are organized in a system of $2N$ non-linear equations of second order in $2N$ unknown functions of time. How can these equations be solved with Euler's method? The answer is contained in the previous three sections of this appendix. Indeed, using the same mathematical trick used in the previous section, it is easy to understand how the equations (8.10) can be transformed into a system of $4N$ differential equations of the first order in $4N$ unknown functions of time. These unknown functions will be the nodes' coordinates of the model and the relative components of velocity. Therefore, the resolution of the equations (8.10) can easily be obtained using the Euler method, as already explained.

• ***The Euler Method: Final Considerations***

Even if the Euler method is better than the finite differences method explained in chapter 9, it still isn't particularly efficient. But what exactly does it mean to say that a numerical method isn't "efficient"? The question is perfectly sensible.

Obviously, before we can say that a particular numerical technique isn't efficient, we need another numerical technique to compare it to. So it would be more correct to state that the Euler method isn't as efficient as other numerical methods. But saying shifts the problem. Now the question is: what exactly is meant by the word "efficiency"? The answer isn't trivial for the simple reason that the concept of "efficiency" depends greatly on the purpose of our use of a numerical technique.

Usually the concept of efficiency of a numerical method is strictly related to both the accuracy of the results and the speed of calculus. Depending on our particular purpose, it could sometimes be necessary to compromise either the accuracy of the results or the speed of the calculus. Moreover, for certain kinds of calculus under the category of "simulations", the purposes are both accuracy of results and speed of calculus; but this isn't an iron rule. Indeed, although the sodaconstructor is essentially a simulation program, it is important to remember that it is also just a game[36], so it isn't particularly important to obtain perfect accuracy of the

[36] A beautiful game in my opinion.

results. It is enough that the models can walk reliably on the screen. Therefore, for the sodaconstructor, it might be better to favor the speed of calculus.
The Euler method isn't particularly efficient with either accuracy or speed. Still, this method is good enough for the sodaconstructor's purposes[37].
The last question is: what numerical methods are better than Euler?

• ***The Runge-Kutta's Algorithm***

Here I wish to say just a few words about the Runge Kutta algorithm. The main reason is that this algorithm was used in another simulation game similar to the sodaconstructor but in 3D: Beaker's Springs Lab.
The first important thing to say is that the Runge-Kutta algorithm is based on the Euler method. Also, the Runge-Kutta algorithm represents a class of numerical techniques. For this reason, here I'll talk only about the first order Runge-Kutta method[38].
The problem that we'll take into consideration is defined by the following non-linear differential equation of the first order in normal form:

$$y'(x) = f[y(x), x] \tag{C.18}$$

together with the boundary condition:

$$y(x_0) = y_0 \tag{C.19}$$

The basic idea of the Runge-Kutta first order method is to use a different slope in order to get a more accurate value of the approximate value y_1. In the Euler method the value of the slope was:

$$s_0 = y'(x_0) = f(y_0, x_0) \tag{C.20}$$

[37] i.e. our entairtment.
[38] The zero order Runge-Kutta method is just the Euler method.

In the Runge Kutta method, however, the value of the slope is taken in the middle of the interval $(x_0, x_0 + h)$. As can be seen in Figure C.2, the value of this slope is taken at the coordinates $x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}s_0 h$ where s_0 is the value defined by (C.20). This new and more accurate slope $\tilde{s}_0$ is given by (C.18) by means of the relation:

$$\tilde{s}_0 = y'\left(x_0 + \tfrac{1}{2}h\right) \cong f\left(y_0 + \tfrac{1}{2}s_0 h, x_0 + \tfrac{1}{2}h\right) \tag{C.21}$$

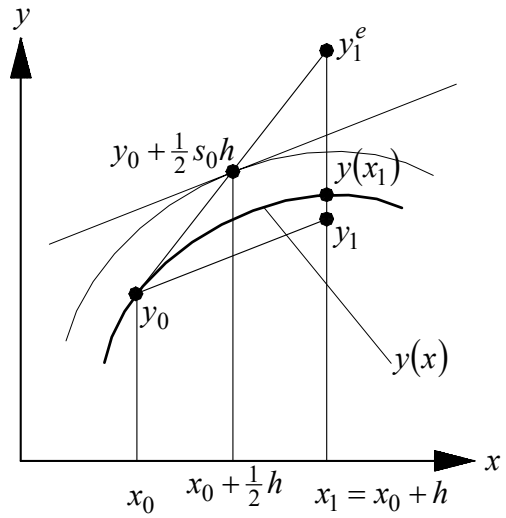

Figure C.2: The Runge-Kutta's method.

Therefore the approximate value of the function $y(x)$ in correspondence with the abscissa $x_1 = x_0 + h$ is given by:

$$y_1 = y_0 + \tilde{s}_0\, h \tag{C.22}$$

It is clear from Figure C.2 that this approximate value is more accurate than the corresponding value y_1^e of the Euler method.

At this point, it is possible to find the sequence of points of our unknown function $y(x)$ by simply iterating the above procedure.

The generic formulas for step number *i* of the above procedure are the following:

$$\begin{cases} x_i = x_0 + i\,h \\ s_{i-1} = f\left(y_{i-1}, x_{i-1}\right) \\ \widetilde{s}_{i-1} = f\left(y_{i-1} + \frac{1}{2}\,s_{i-1}h, x_{i-1} + \frac{1}{2}\,h\right) \\ y_i = y_{i-1} + \widetilde{s}_{i-1}\,h \end{cases} \tag{C.23}$$

It is possible to use the Runge-Kutta first order method for different objectives. The first possibility is to use this numerical method in order to obtain an approximate solution more accurate than the solution obtained using the Euler method, while still maintaining the same computational charge. Or, a second possibility is to use this method in order to get an approximate solution as accurate as the solution obtained using the Euler method, but having a smaller computational charge.