Effective bounds on strong unicity in L_1-approximation

U. Kohlenbach and P. Oliva

BRICS
University of Århus
Denmark

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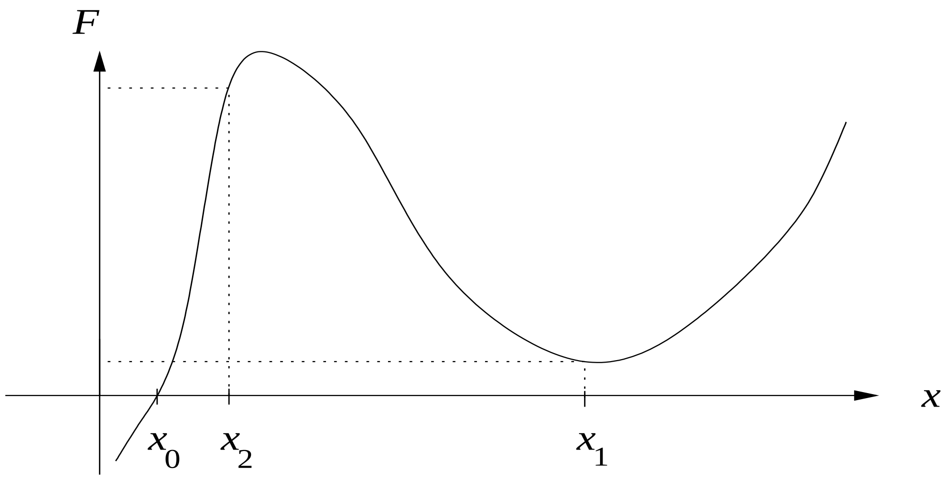
The problem of computing a root

Given that a continuous $F : \mathbb{R} \to \mathbb{R}$ has a unique root (let x_0 be the root and $\varepsilon \in \mathbb{Q}_+^*$):

(i) Find ε-roots, i.e. x_1 such that $|F(x_1)| < \varepsilon$.

(ii) Find x_2 ε-close to x_0, i.e. $|x_2 - x_0| < \varepsilon$.



- We consider here a problem of type (ii), i.e. we look for an algorithm $B : \mathbb{Q}_+^* \to \mathbb{Q}$ such that $|B(\varepsilon) - x_0| < \varepsilon$. We say that B computes the root x_0.

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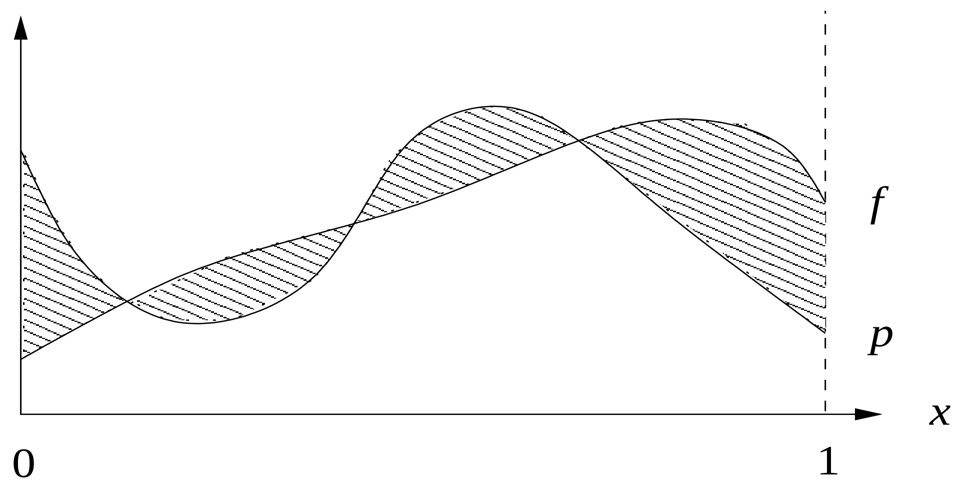
Our function F (L_1-approximation)

- Fix $f \in C[0,1]$ and $n \in \mathbb{N}$.
- $P_n :=$ polynomials of degree $\leq n$.
- L_1-distance, $\|f - p\|_1 := \int_0^1 |f(x) - p(x)| dx$.



- $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$.
- The problem: $F(p) := \|f - p\|_1 - dist_1(f, P_n)$.

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Existence of the best L_1-approximant

Theorem: For any $f \in C[0,1]$ and $n \in \mathbb{N}$, there exists a $p_n \in P_n$ such that $F(p_n) = 0$.

- The existence proof is non-constructive (compactness argument) and does not supply an algorithm B for computing a root of F.
- The roots of F must belong to a compact set, e.g.,

$$K_{f,n} := \{p \in P_n : \|p\|_1 \leq 2\|f\|_1\}.$$

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Uniqueness of the best L_1-approximant

Theorem [Jackson, 1921]: For any $f \in C[0,1]$ and $n \in \mathbb{N}$ the best L_1-approximation of f from P_n is unique.

In logical terms:

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in P_n (\bigwedge_{i=1}^{2} (F(p_i) = 0) \to \|p_1 - p_2\|_1 = 0).$$

which is equivalent to,

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}$$

$$(\bigwedge_{i=1}^{2} (F(p_i) = 0) \to \|p_1 - p_2\|_1 = 0),$$

and hence to

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; k \in \mathbb{N}\ \exists l \in \mathbb{N}$$

$$(\bigwedge_{i=1}^{2} (|F(p_i)| \leq 2^{-l}) \to \|p_1 - p_2\|_1 < 2^{-k}).$$

Extracting the uniform modulus of uniqueness

From the uniqueness proof (even though it is non-constructive) we can extract information for computing the best L_1-approximation of a given function.

Cheney's proof [1965] of Jackson's theorem can be formalized in $\mathbf{PA}^\omega + AC_{qf} + WKL$ [Kohlenbach, 90]. From that proof we extract a Φ s.t.

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; k \in \mathbb{N}$$
$$(\bigwedge_{i=1}^{2}(|F(p_i)| \leq 2^{-\Phi(f,n,k)}) \to \|p_1 - p_2\|_1 < 2^{-k}).$$

AC_{qf} : quantifier free axiom of choice.

WKL : binary ('weak') König lemma.

The metatheorem

Theorem [Kohlenbach, 1993]: Let X, K be **PA**$^{\omega}$-definable Polish spaces, K compact and consider a sentence which can be written (when formalized in the language of **PA**$^{\omega}$) in the form

$$A := \forall x \in X; n \in \mathbb{N}; y \in K \; \exists k \in \mathbb{N} \, A_1(n, x, y, k),$$

where A_1 is a purely existential. Then the following rule holds:

$$\left\{ \begin{array}{l} \textbf{PA}^{\omega} + AC_{qf} + WKL \vdash \forall x \in X; n \in \mathbb{N}; y \in K \; \exists k \in \mathbb{N} \, A_1(n, x, y, k) \\ \text{then one can extract a primitive recursive (in the sense of} \\ \text{Gödel'58) term } \Phi \text{ s.t.} \\ \textbf{HA}^{\omega} \vdash \forall x \in X; n \in \mathbb{N}; y \in K \exists k \leq \Phi(n, x) \, A_1(n, x, y, k). \end{array} \right.$$

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The uniform modulus of uniqueness

Let

$$\Phi(\omega, n, \varepsilon) := \min\{\tfrac{c_n\varepsilon}{3^{n+2}(n+1)^{n+1}}, \tfrac{c_n\varepsilon}{2}\omega_n(\tfrac{c_n\varepsilon}{2})\},\ where$$
$$c_n := \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{n+3}3^{n^2+2n}(n+1)^{n^2+2n+1}}\ and$$
$$\omega_n(\varepsilon) := \min\{\omega(\tfrac{\varepsilon}{4}), \tfrac{\varepsilon}{40(n+1)^4\lceil\frac{1}{\omega(1)}\rceil}\}.$$

The functional Φ is a uniform modulus of uniqueness for the best L_1-approximation of any function f in $C[0,1]$ (having modulus of uniform continuity ω) from P_n, i.e.

$$\forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+$$
$$\Big(\bigwedge_{i=1}^{2}(|F(p_i)| \leq \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 < \varepsilon\Big).$$

- Φ has optimal ε-dependency!

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Computing the best L_1-approximation

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+$$
$$\big(\bigwedge_{i=1}^{2} (|F(p_i)| \leq \Phi(\omega_f, n, \varepsilon)) \to \|p_1 - p_2\|_1 < \varepsilon\big).$$

Take p_1 to be the best L_1-approximation of f from P_n (say p_n),

$$\forall f \in C[0,1]; n \in \mathbb{N}; p \in P_n; \varepsilon \in \mathbb{Q}^*_+$$
$$\big(|F(p)| \leq \Phi(\omega_f, n, \varepsilon) \to \|p - p_n\|_1 < \varepsilon\big).$$

We only need Ψ such that:

$$\forall f \in C[0,1]; n \in \mathbb{N}; \varepsilon \in \mathbb{Q}^*_+ \big(\Psi(f, n, \varepsilon) \in P_n \wedge |F(\Psi(f, n, \varepsilon))| < \varepsilon\big)$$

Then,

$$B_{f,n}(\varepsilon) := \Psi(f, n, \Phi(\omega_f, n, \varepsilon))$$

computes the best L_1-approximation of f from P_n.

Conclusions

- 1921 [Jackson] Proof of the uniqueness of best L_1-approximation.
- 1965 [Cheney] 'Elementary' proof of Jackson's theorem (highly non-constructive, WKL).
- 1975 [Björnestål] Existence of a uniform modulus of uniqueness of the form $c\,\varepsilon\,\omega_f(c\,\varepsilon)$, where the constant c depends on f and n (but the constant c is not produced).
- 1979 [Kroó] Björnestål's result is improved: the constant c need not to depend on any particular value of f but only on its modulus of uniform continuity ω_f (again the constant c is not produced). Moreover, Björnestål's ε-dependency is optimal.
- 2001 [Koh,Oli] Fully explicit uniform modulus of uniqueness.
- In progress: First complexity upper bound for p_n.

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