Functional Interpretations and Games

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Previously...

- Unifying framework for functional interpretations (*Dialectica*, *Diller-Nahm*, *mod. realizability*)

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- Flexibility on treatment of $A \rightarrow B$

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- Hyland, de Paiva, Blass, Shirahata

Outline

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Mathematics is like a game, mathematicians are always winners because they play both roles

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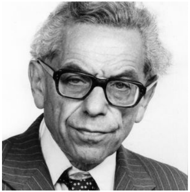
View a mathematical statement as the description of a game

$$\forall n \geq 2\, \exists x, y, z \left(\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$



Paul Erdös

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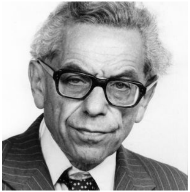

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$$f_0, f_1, f_2 : \mathbb{N} \rightarrow \mathbb{N}^*$$

$$n \in \{2, \ldots\}$$

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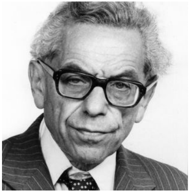

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$$f_0, f_1, f_2 : \mathbb{N} \to \mathbb{N}^* \qquad n \in \{2, \ldots\}$$

$$\frac{4}{n} = \frac{1}{f_0(n)} + \frac{1}{f_1(n)} + \frac{1}{f_2(n)}$$

Games: Formal Description

- **Game**

  $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$

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  Eloise and Abelard

- **Two domains of moves**

  $x \in D_1$ and $y \in D_2$

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- **Adjudication of Winner**

  Relation $R(x, y)$ between players' moves

  (*usually* $|G|^x_y$)

Games: Examples

Domain 1	Domain 2	Adjudication
$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y \bmod 3$

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

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$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$
$f_i \in \mathbb{N} \to \mathbb{N}^*$	$n \geq 2$	$\frac{4}{n} = \frac{1}{f_0 n} + \frac{1}{f_1 n} + \frac{1}{f_2 n}$

Goal

A is true (is provable)

iff

Eloise has winning move in game $|A|^x_y$

Symmetry

Game $\neg A$ should be game A with roles reversed

$$|\neg A|_y^x \equiv \neg |A|_x^y$$

$$|\neg\neg A|_y^x \equiv |A|_y^x$$

Interpretation

$$|A \wedge B|^{x,v}_{f,g} \quad :\equiv \quad |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

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$$|\forall z A(z)|^{f}_{y,z} \quad :\equiv \quad |A(z)|^{fz}_{y}$$

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$$|\exists z A(z)|^{x,z}_{f} \quad :\equiv \quad |A(z)|^{x}_{fz}$$

Interpretation

$$|A \otimes B|^{x,v}_{f,g} \quad :\equiv \quad |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

$$|A \multimap B|^{f,g}_{x,w} \quad :\equiv \quad |A|^{x}_{fw} \text{ implies } |B|^{gx}_{w}$$

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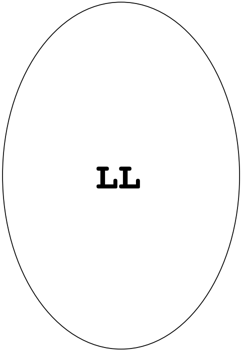
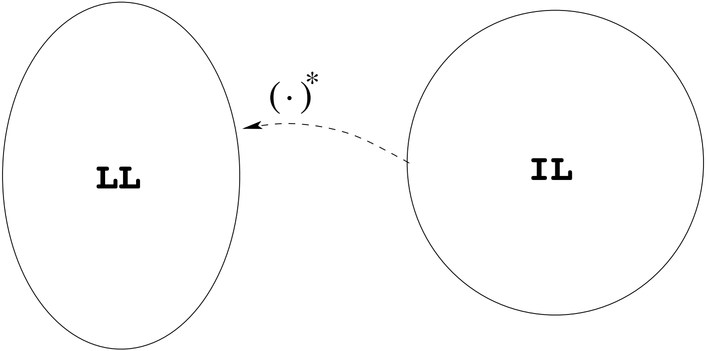
$$|\exists z A(z)|^{x,z}_{f} \quad :\equiv \quad |A(z)|^{x}_{fz}$$

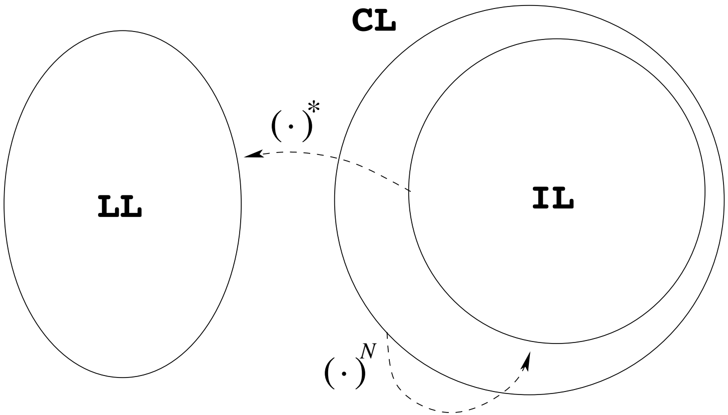
Linear Logic



Linear Logic



Linear Logic



Linear Logic: Duality

$$
\begin{aligned}
(A_{\mathsf{at}})^{\perp} &\equiv A_{\mathsf{at}}^{\perp} & (\exists z A)^{\perp} &\equiv \forall z A^{\perp} \\
(A_{\mathsf{at}}^{\perp})^{\perp} &\equiv A_{\mathsf{at}} & (\forall z A)^{\perp} &\equiv \exists z A^{\perp} \\
(A \parr B)^{\perp} &\equiv A^{\perp} \otimes B^{\perp} & (?A)^{\perp} &\equiv !(A^{\perp}) \\
(A \otimes B)^{\perp} &\equiv A^{\perp} \parr B^{\perp} & (!A)^{\perp} &\equiv ?(A^{\perp})
\end{aligned}
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$$

$$A \multimap B \equiv A^{\perp} \parr B$$

Linear Logic: Structural

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{(cut)} \qquad A \vdash A \quad \text{(id)}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, B^{\perp} \vdash A^{\perp}}\ (\perp) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\ \text{(per)}$$

Linear Logic: Connectives and Quantifiers

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall z^{\rho} A}\,(\forall) \qquad \frac{\Gamma \vdash A[t^{\rho}/z]}{\Gamma \vdash \exists z^{\rho} A}\,(\exists)$$

Linear Logic: Modalities

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash ?A}\ (?)$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{(con)} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{(wkn)}$$

Exponential Games

$$!A \qquad\qquad ?A$$

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$$\begin{aligned} (1) \quad |!A|^x &:\equiv \forall y |A|^x_y \\ |?A|_y &:\equiv \exists x |A|^x_y \end{aligned}$$

Exponential Games

$$!A \qquad ?A$$

$$
\begin{aligned}
(1) \quad & |!A|^x & :\equiv \; & \forall y |A|^x_y \\
& |?A|_y & :\equiv \; & \exists x |A|^x_y \\
(2) \quad & |!A|^x_f & :\equiv \; & \forall y \in f x \, |A|^x_y \\
& |?A|^f_y & :\equiv \; & \exists x \in f y \, |A|^x_y
\end{aligned}
$$

Exponential Games

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(2) \quad & |!A|^x_f & :\equiv \quad & \forall y \in fx \, |A|^x_y \\
& |?A|^f_y & :\equiv \quad & \exists x \in fy \, |A|^x_y \\
(3) \quad & |!A|^x_f & :\equiv \quad & |A|^x_{fx} \\
& |?A|^f_y & :\equiv \quad & |A|^{fy}_y
\end{aligned}
$$

LL $\Rightarrow$ G

Theorem

If

$\Gamma \vdash A$ *is provable in linear logic*

then

Eloise wins game $\Gamma \multimap A$

LL $\Rightarrow$ G

Theorem

If

$\Gamma \vdash A$ *is provable in linear logic*

then

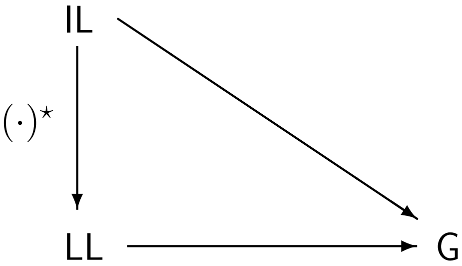
Eloise wins game $\Gamma \multimap A$ *, i.e.*

she has moves t, r *such that for all* v, y

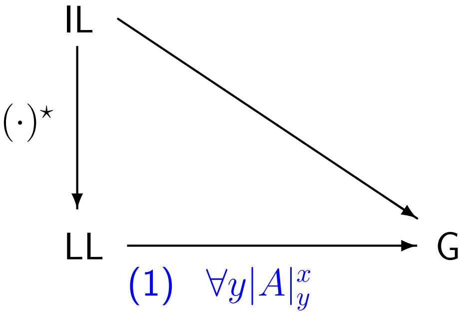
$$|\Gamma|^{v}_{r(y)} \vdash |A|^{t(v)}_{y}$$

Outline

1. Games
2. Functional Interpretations
3. Characterisation

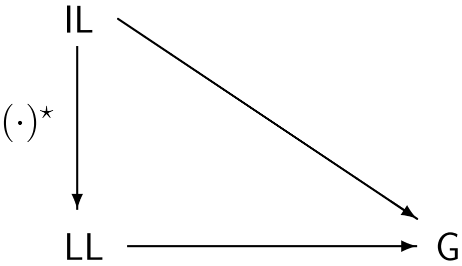

IL
$(\cdot)^\star$
LL
G

Modified realizability (Kreisel'1959)



Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



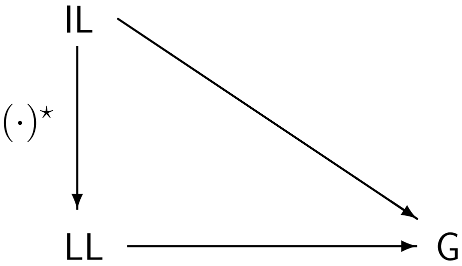
(1) $\forall y |A|_y^x$

(2) $\forall y \in f x \, |A|_y^x$

Dialectica interpretation (Gödel'1958)

Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



(1) $\forall y |A|^x_y$

(2) $\forall y \in f x \, |A|^x_y$

(3) $|A|^x_{fx}$

Parametrised Interpretation

Bounded quantifiers $\forall x^{\rho} \sqsubset a^{\rho^*}\, A$ and $\exists x^{\rho} \sqsubset a^{\rho^*}\, A$

$$
\begin{aligned}
(\forall x \sqsubset a\, A)^{\perp} &\equiv \exists x \sqsubset a\, A^{\perp} \\
(\exists x \sqsubset a\, A)^{\perp} &\equiv \forall x \sqsubset a\, A^{\perp}
\end{aligned}
$$

Parametrised interpretation:

$$
\begin{aligned}
|!A|^{x}_{f} &:\equiv \forall y \sqsubset f x\, |A|^{x}_{y} \\
|?A|^{f}_{y} &:\equiv \exists x \sqsubset f y\, |A|^{x}_{y}
\end{aligned}
$$

Parametrised Interpretation

$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$	$c_A \;:\; \rho^*$
$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$	$\varepsilon_A \;:\; \rho^* \times \rho^* \hookrightarrow \rho^*$
$\dfrac{\Gamma \vdash A}{\Gamma \vdash ?A}$	$\eta_A \;:\; \rho \hookrightarrow \rho^*$
$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$	$\mu_A \;:\; \rho \to \tau^* \hookrightarrow \rho^* \to \tau^*$

Outline

Characterisation (modified realizability)

$$
\begin{array}{lcl}
|A \otimes B|^{x,v}_{f,g} & :\equiv & |A|^{x}_{fv} \otimes |B|^{v}_{gx} \\
|A \multimap B|^{f,g}_{x,w} & :\equiv & |A|^{x}_{fw} \multimap |B|^{gx}_{w} \\
|\forall z A(z)|^{f}_{y,z} & :\equiv & |A(z)|^{fz}_{y} \\
|\exists z A(z)|^{x,z}_{f} & :\equiv & |A(z)|^{x}_{fz} \\
|!A|^{x} & :\equiv & !\forall y |A|^{x}_{y} \\
|?A|_{y} & :\equiv & ?\exists x |A|^{x}_{y}
\end{array}
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Characterisation (modified realizability)

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

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Characterisation (modified realizability)

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

axiom

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

Characterisation (modified realizability)

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

axiom

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

Characterisation (modified realizability)

$$|\Gamma|^{v}_{r[y]} \vdash |A|^{t[v]}_{y}$$

axiom

cut rule

$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

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Characterisation (modified realizability)

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axiom

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cut rule

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

$$\exists\!\!\forall^{v}_{w} |\Gamma|^{v}_{w} \vdash \exists\!\!\forall^{x}_{y} |A|^{x}_{y}$$

Simultaneous Quantifier

$$\frac{A_0(a_0, y_0), \ldots, A_n(a_n, y_n)}{\exists\!\!\!/^{x_0}_{y_0} A_0(x_0, y_0), \ldots, \exists\!\!\!/^{x_n}_{y_n} A_n(x_n, y_n)} \; (\exists\!\!\!/)$$

y_i may only appear free in the terms a_j, for $j \neq i$

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y_i may only appear free in the terms a_j, for $j \neq i$

$$\frac{(x = y), (x \neq y)}{\exists\!\!\!/\,^{x}_{y}(x = y), \exists\!\!\!/\,^{y}_{x}(x \neq y)} \; (\exists\!\!\!/\,)$$

New Principles

- Sequential choice

  $\forall z \exists\!\!\!\forall^{x}_{y} A(x, y, z) \multimap \exists\!\!\!\forall^{f}_{y,z} A(fz, y, z)$

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- Sequential choice

$$\forall z \exists\!\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\!\forall^f_{y,z} A(fz, y, z)$$

- Parallel choice

$$\exists\!\!\!\forall^x_y A(x) \parr \exists\!\!\!\forall^v_w B(v) \multimap \exists\!\!\!\forall^{f,g}_{y,w} (A(fw) \parr B(gy))$$

New Principles

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  $\exists\!\!\!\forall^{x}_{y} A(x) \otimes \exists\!\!\!\forall^{v}_{w} B(v) \multimap \exists\!\!\!\forall^{f,g}_{y,w} (A(fw) \otimes B(gy))$

- Trump advantage

  $!\exists\!\!\!\forall^{x}_{y} A \multimap \exists x !\forall y A$

New Principles

- Sequential choice

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- Trump advantage

  $!\exists\!\!\forall^x_y A \multimap \exists x ! \forall y A$

Theorem

These are sufficient for deriving the equivalence between A and its interpretation $\exists\!\!\forall^x_y |A|^x_y$.

Summary

- Functional interpretations of linear logic

  Usual interpretations of IL derivable

- Interesting use of (simple) branching quantifier

  Characterisation using branching quantifier

- Sound extensions of linear logic