Modified Realizability Interpretation of Classical Linear Logic

Paulo Oliva

Queen Mary, University of London, UK
(pbo@dcs.qmul.ac.uk)

LICS, Wrocław, Poland
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THE ROYAL SOCIETY

Outline

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Formula $A \quad \Rightarrow \quad$ **Set of functionals** $|A|$

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Proof π	$\Rightarrow$	**Functional** $f_\pi \in \lvert A \rvert$
$\ldots$		$\lambda x . x + 1$
$S_1 \vdash_\pi A$	$\Rightarrow$	$S_2 \vdash f_\pi \in \lvert A \rvert$

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Modified realizability (Kreisel 1959)

- *Independence results* for HA

$$\begin{cases} |P| \text{ set of non-computable functionals} \\ \mathsf{HA} \vdash P \quad \Rightarrow \quad \mathsf{HA} \vdash \exists f (f \in |P|) \end{cases}$$

Functional Interpretations

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  $S + P \vdash A \quad \Rightarrow \quad S \vdash A$

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- *Independence results*

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- *Conservation results*

  $S + P \vdash A \quad \Rightarrow \quad S \vdash A$
- *Proof mining*

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- *Build models*

  $\mathcal{M} \models S$

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- What about modified realizability of LL?
- Observation of Blass (1995) provides the answer

Linear Logic (Girard 1987)

- Explicit treatment of **contraction**

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \quad \Rightarrow \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

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- **Refinement** of intuitionistic implication

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- **Refinement** of logical connectives

	conjunction	disjunction
additive	$\wedge$	$\vee$
multiplicative	$\otimes$	$\parr$

Outline

Games: Formal Description

- Game $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$
- **Two players**

  Eloise and Abelard
- **Two domains of moves**

  $x \in D_1$ and $y \in D_2$
- **Adjudication of Winner**

  Relation $R(x, y)$ between players' moves

  (*usually* $|G|^x_y$)

Games: Examples

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

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Goal

A is true (is provable)

iff

Eloise has winning move in game $|A|^x_y$

Symmetry

Game $A^\perp$ should be game A with roles reversed

$$|A^\perp|^x_y \equiv \neg |A|^y_x$$

$$|(A^\perp)^\perp|^x_y \equiv |A|^x_y$$

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$$|\forall z A(z)|^{f}_{y,z} \quad :\equiv \quad |A(z)|^{fz}_{y}$$

$$|\exists z A(z)|^{x,z}_{f} \quad :\equiv \quad |A(z)|^{x}_{fz}$$

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$$\begin{cases} |!A|^x_f & :\equiv & |A|^x_{fx} \\ |?A|^f_y & :\equiv & |A|^{fy}_y \end{cases}$$

(2) unlimited set of moves

$$\begin{cases} |!A|^x & :\equiv & \forall y |A|^x_y \\ |?A|_y & :\equiv & \exists x |A|^x_y \end{cases}$$

Soundness

Theorem

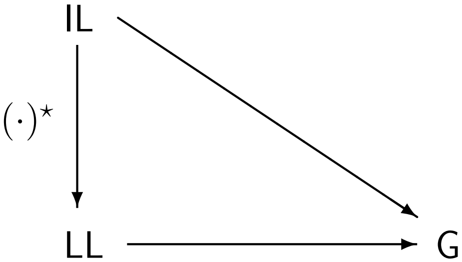
If

$\Gamma \vdash A$ *is provable in linear logic*

then

Eloise wins game $\Gamma \multimap A$.

Relation to Interpretation of IL



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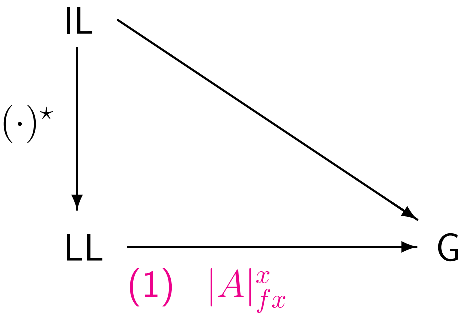
Dialectica interpretation (Gödel'1958)



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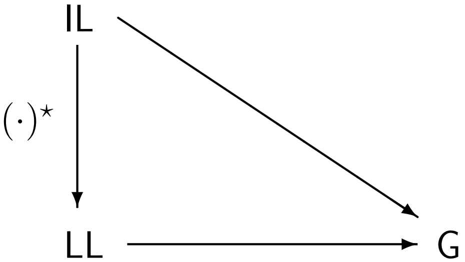
Modified realizability (Kreisel'1959)



(1) $|A|^x_{fx}$

(2) $\forall y |A|^x_y$

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- What about the other way around?
- For which extension of LL do we have the converse?

Characterisation

$$A$$

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$$A$$

$$\swarrow$$

$$\exists x \forall y |A|^x_y$$

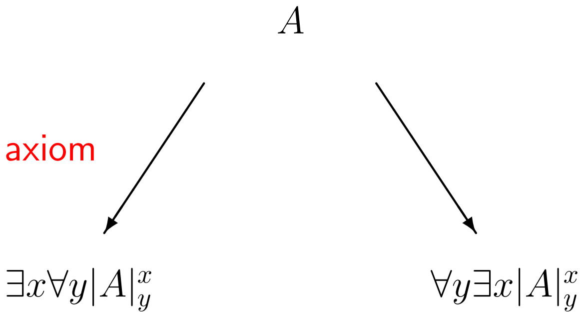
Characterisation

A

axiom

$$\exists x \forall y |A|^x_y$$

Characterisation



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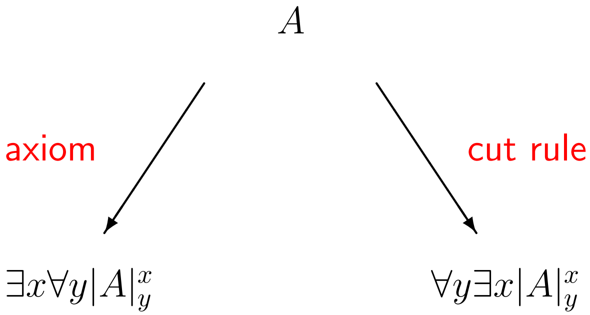

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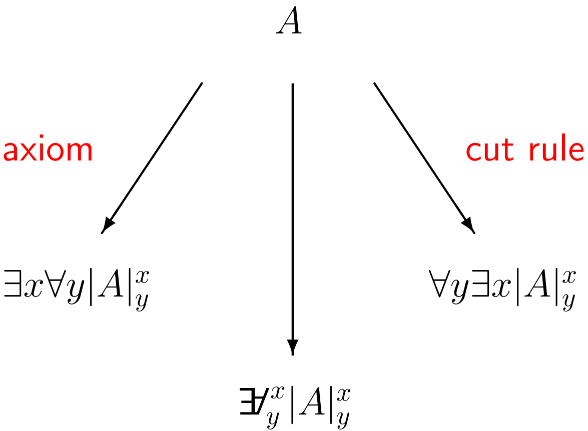

New Principles

- **Sequential choice**

  $$\forall z \exists\!\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\!\forall^f_{y,z} A(fz, y, z)$$

- **Parallel choice**

  $$\exists\!\!\!\forall^x_y A(x) \parr \exists\!\!\!\forall^v_w B(v) \multimap \exists\!\!\!\forall^{f,g}_{y,w}(A(fw) \parr B(gy))$$

- **Trump advantage**

  $$!\exists\!\!\!\forall^x_y A \multimap \exists x ! \forall y A$$

New Principles

Theorem

These principles are sufficient for deriving the equivalence between A and its interpretation $\exists\!\!\!\forall^x_y |A|^x_y$.

Summary

- **Modified realizability** of classical linear logic

  M.r. interpretation of IL derivable

- Interesting use of (simple) **branching quantifier**

  Characterisation using branching quantifier

- Sound **extensions** of linear logic

  Conservation results

  Stronger disjunction and existence properties

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