Variants of Modified Bar Recursion

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Summary

Better way of understanding modified bar recursion

(*via selection functionals*)

Issues of efficiency

(*in case we ever need bar recursion in practise*)

Outline

1. Introduction
   - Role of contraction
   - Dialectica interpretation
2. Modified Bar Recursion
   - Selection functions
   - BBC functional
3. Other Variants
   - Berger
   - Escardo
4. Conclusions

Outline

Importance of contraction

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$$\begin{array}{lcl} Kxy & \mapsto & x \\ Sxyz & \mapsto & xz(yz) \end{array}$$

Importance of contraction

$$Kxy \mapsto x$$

$$Sxyz \mapsto xz(yz)$$

Importance of contraction

$$Kxy \quad \mapsto \quad x \qquad \text{(weakening)}$$

$$Sxyz \quad \mapsto \quad xz(yz) \qquad \text{(contraction)}$$

Importance of contraction

Herbrand theorem: if $\exists x A(x)$ then $\bigvee A(t_i)$

Cut elimination: cut rule is admissible

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An *elimination of contractions* procedure

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$(\lambda x.t[x])r \ \mapsto \ t[r]$

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Cut elimination: cut rule is admissible

$(\lambda x.t[x])r \;\mapsto\; t[r]$

$(\lambda x.t[x, x])r \;\mapsto\; t[r, r]$

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$(\lambda x.t[x,x])r \;\mapsto\; (\lambda x_0 \lambda x_1.t[x_0,x_1])rr$

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Becomes an *elimination of contractions* procedure

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What are negative translations useful for?

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How do they do it?

Move contractions from the conclusion to the premise

Classical theorem: $A \wedge B, \neg A \vee \neg B$

$$\dfrac{\dfrac{\dfrac{\dfrac{A, \neg A \qquad B, \neg B}{A \wedge B, \neg A, \neg B}\,(\wedge\mathsf{I})}{A \wedge B, \neg A \vee \neg B, \neg B}\,(\vee\mathsf{I})}{A \wedge B, \neg A \vee \neg B, \neg A \vee \neg B}\,(\vee\mathsf{I})}{A \wedge B, \neg A \vee \neg B}\,(\mathsf{con})$$

Classical theorem: $A \wedge B, \neg A \vee \neg B$

$$
\cfrac{\cfrac{\cfrac{\cfrac{A, \neg A \qquad B, \neg B}{A \wedge B, \neg A, \neg B}\ (\wedge I)}{A \wedge B, \neg A \vee \neg B, \neg B}\ (\vee I)}{A \wedge B, \neg A \vee \neg B, \neg A \vee \neg B}\ (\vee I)}{A \wedge B, \neg A \vee \neg B}\ (\text{con})
$$

Intuitionistic version: $\neg(\neg A \vee \neg B) \to \neg\neg(A \wedge B)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(A \wedge B)]_\delta \qquad \dfrac{[A]_\alpha \quad [B]_\beta}{A \wedge B}}{\bot}}{\neg A}\,(\alpha)}{\neg A \vee \neg B} \qquad \neg(\neg A \vee \neg B)}{\bot}}{\neg B}\,(\beta)}{\neg A \vee \neg B} \qquad \neg(\neg A \vee \neg B)}{\bot}}{\neg\neg(A \wedge B)}\,(\delta)
$$

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$$

Key principle

$$\neg(\neg A \vee \neg B) \rightarrow \neg\neg(A \wedge B)$$

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... and using induction ....

$$\neg\exists b \leq n \neg A(b) \rightarrow \neg\neg\forall b \leq n A(b)$$

Example

Infinite pigeonhole principle

$$\forall n \forall f^{\mathbb{N}\to n} \exists b \leq n \underbrace{\forall x \exists y > x (fy = b)}_{\{y \,:\, fy=b\} \text{ infinite}}$$

Follows (classically) from BC *for* Π^0_1*-formulas.*

Between Σ^0_2 *and* Σ^0_1 *induction.*

Infinitary form

What about

$$\neg \forall n A(n) \to \exists n \neg A(n)$$

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Infinite number of contractions.

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Corresponds to infinite number of LEM applications

... as with comprehension functions

$$\exists f \forall n (fn = 0 \leftrightarrow A(n))$$

Informally

How do we deal with infinitely many applications?

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How do we deal with infinitely many applications?

In practise, only a finitary portion of that is used!

Interpret using Dialectica

Dialectica interpretation of DNS

$$\neg \exists n \neg A(n) \rightarrow \neg\neg \forall n A(n)$$

leads to a set of equations (on Ψ, Φ, Δ)

$$\begin{aligned} n &\stackrel{\mathbb{N}}{=} \Psi f \\ f_n &\stackrel{\rho}{=} \Phi_n g_n \\ g_n(f_n) &\stackrel{\tau}{=} \Delta f \end{aligned}$$

Possible to solve (no need for all solutions f)

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What about a direct interpretation (realizability)?

Outline

Axiom of choice

$$\forall x^{\tau} \exists y^{\rho} A(x, y) \to \exists f^{\tau \to \rho} \forall x A(x, fx)$$

Equivalent to:

the Cartesian product of an arbitrary
collection of non-empty sets is non-empty

Axiom of **countable** choice

$$\forall x^{\mathbb{N}} \exists y^{\rho} A(x, y) \rightarrow \exists f^{\mathbb{N} \rightarrow \rho} \forall x A(x, fx)$$

Equivalent to:

the Cartesian product of a **countable** collection of non-empty sets is non-empty

Selection functions

Definition (Escardo'07)

A computable functional

$$\Psi \quad : \quad (A \to \mathbb{B}) \to A$$

is called a *selection functional* for A if for any predicate

$$Y \quad : \quad A \to \mathbb{B}$$

$\Psi(Y) \in Y$ whenever Y is not empty.

Selection functions

Problem: Given a family of selection functions

$$\Phi_n \; : \; (A(n) \to \mathbb{B}) \to A(n)$$

how do we produce a selection function

$$\Psi \; : \; (\forall n A(n) \to \mathbb{B}) \to \forall n A(n)$$

for the product?

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for the product? Define

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x^{A(n)}.\overbrace{Y(\underbrace{\Psi_Y(s * \langle n, x \rangle)}_{\forall n A(n)})}^{\mathbb{B}})$$

Assume continuity and take $\Psi_Y()$.

(General) selection functions

Problem: Given a family of (general) selection functions

$$\Phi_n \ : \ (A(n) \to \mathbb{N}) \to A(n)$$

how do we produce a (general) selection function

$$\Psi \ : \ (\forall n A(n) \to \mathbb{N}) \to \forall n A(n)$$

for the product? Define

$$\Psi_Y(s) = s \mathbin{@} \lambda n. \Phi_n(\lambda x^{A(n)}. \overbrace{Y(\underbrace{\Psi_Y(s * \langle n, x \rangle)}_{\forall n A(n)})}^{\mathbb{N}})$$

Assume continuity and take $\Psi_Y()$.

DNS

Has exactly the type of DNS

$$\neg\exists n \neg A(n) \to \neg\neg\forall n A(n)$$

i.e.

$$\forall n(\underbrace{\neg A(n) \to A(n)}_{\Phi_n}) \to \underbrace{\neg\forall n A(n)}_{Y} \to \forall n A(n)$$

BBC functional

$$\Psi_Y(s) = s\,@\,\lambda n.\Phi_n(\lambda x^{A(n)}.\overbrace{Y(\underbrace{\Psi_Y(s * \langle n, x\rangle)}_{\forall n A(n)})}^{\perp})$$

Outline

Possibilites

Option 1 (BBC)

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(s * \langle n, x \rangle)))$$

Option 2 (U. Berger)

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(s * \langle |s|, x \rangle)))$$

Option 3 (M. Escardo)

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(\overline{\Psi_Y(s)}(n) * \langle n, x \rangle)))$$

BBC functional

$$\Psi_Y(s) = s \mathbin{@} \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(s * \langle n, x \rangle)))$$

- Efficient
- **Not** easy to prove total
- **Not** easy to prove it is a realiser

Berger's functional

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(s * \langle |s|, x\rangle)))$$

- **Not** very efficient
- Easy to prove total
  (*by bar induction*)
- Easy to prove it is a realiser
  (*by bar induction*)

Escardo's functional

$$\Psi_Y(s) = s \,@\, \lambda n.\Phi_n(\lambda x.Y(\Psi_Y(\overline{\Psi_Y(s)}(n) * \langle n, x \rangle)))$$

- Efficient
- Easy to prove total
  (*by course-of-value bar induction*)
- Easy to prove it is a realiser
  (*by course-of-value bar induction*)

Definability

Theorem

Escardo's is primitive recursively definable in Berger's

Definability

Theorem

Escardo's is primitive recursively definable in Berger's

Other connections still open!

Outline

Summary

- Motivation of modified bar recursion via selection functions
- Three variants of modified bar recursion
- Issues of efficiency and easiness of totality proof

References

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