Realizability Interpretations of Linear Logic

Paulo Oliva

Queen Mary, University of London, UK

(based on joint work with G. Ferreira and J. Gaspar)

Chambéry, 3 June 2009

Outline

1. Realizability (a reformulation)
2. Linear Logic (a model)
3. Functional Interpretations of LL
4. Functional Interpretations of ILL

Outline

Realizability

$$
\begin{array}{rlcl}
\langle x, y \rangle & \mathsf{mr}\ A \wedge B & :\equiv & (x\ \mathsf{mr}\ A) \wedge (y\ \mathsf{mr}\ B) \\
\langle x, y, i \rangle & \mathsf{mr}\ A \vee B & :\equiv & (x\ \mathsf{mr}\ A) \Diamond_i (y\ \mathsf{mr}\ B) \\
f & \mathsf{mr}\ A \to B & :\equiv & \forall x((x\ \mathsf{mr}\ A) \to (fx\ \mathsf{mr}\ B)) \\
\langle x, n \rangle & \mathsf{mr}\ \exists z A & :\equiv & x\ \mathsf{mr}\ A[n/z] \\
f & \mathsf{mr}\ \forall z A & :\equiv & \forall z(fz\ \mathsf{mr}\ A)
\end{array}
$$

where $A \Diamond_i B \ :\equiv\ (i = 0 \to A) \wedge (i = 1 \to B)$.

Realizability

Realizability associates a formula A to a **set** of functionals (e.g. in Gödel's T)

$$S_A \quad :\equiv \quad \{t \ : \ (t \in \mathsf{T}) \wedge (t \ \mathsf{mr} \ A)\}$$

such that A is provable iff S_A is non-empty.

Realizability

Realizability associates a formula A to a **set** of functionals (e.g. in Gödel's T)

$$S_A \;:\equiv\; \{t \;:\; (t \in \mathsf{T}) \wedge (t \;\mathsf{mr}\; A)\}$$

such that A is provable iff S_A is non-empty.

Realizability is a **proof interpretation**:

$$\vdash_\pi A \quad \Rightarrow \quad t_\pi \in S_A$$

Pointwise realizability

Can also be viewed as associating formulas to **relations**

$$
\begin{array}{rlll}
\langle x, v\rangle & \mathsf{pmr}_{y,w} & A \wedge B & :\equiv & (x \ \mathsf{pmr}_y \ A) \wedge (v \ \mathsf{pmr}_w \ B) \\
\langle x, v, i\rangle & \mathsf{pmr}_{y,w} & A \vee B & :\equiv & (x \ \mathsf{pmr}_y \ A) \Diamond_i (v \ \mathsf{pmr}_w \ B) \\
f & \mathsf{pmr}_{x,w} & A \to B & :\equiv & \forall y (x \ \mathsf{pmr}_y \ A) \to (fx \ \mathsf{pmr}_w \ B) \\
\langle x, n\rangle & \mathsf{pmr}_{y} & \exists z A & :\equiv & x \ \mathsf{pmr}_y \ A[n/z] \\
f & \mathsf{pmr}_{z,y} & \forall z A & :\equiv & fz \ \mathsf{pmr}_y \ A.
\end{array}
$$

Pointwise realizability

Can also be viewed as associating formulas to **relations**

$$
\begin{array}{rlllll}
\langle x, v \rangle & \mathsf{pmr}_{y,w} & A \wedge B & :\equiv & (x \ \mathsf{pmr}_y \ A) \wedge (v \ \mathsf{pmr}_w \ B) \\
\langle x, v, i \rangle & \mathsf{pmr}_{y,w} & A \vee B & :\equiv & (x \ \mathsf{pmr}_y \ A) \lozenge_i (v \ \mathsf{pmr}_w \ B) \\
f & \mathsf{pmr}_{x,w} & A \to B & :\equiv & \forall y (x \ \mathsf{pmr}_y \ A) \to (fx \ \mathsf{pmr}_w \ B) \\
\langle x, n \rangle & \mathsf{pmr}_y & \exists z A & :\equiv & x \ \mathsf{pmr}_y \ A[n/z] \\
f & \mathsf{pmr}_{z,y} & \forall z A & :\equiv & fz \ \mathsf{pmr}_y \ A.
\end{array}
$$

An actual realiser refutes all possible challenges.

Lemma

$(x \ \mathsf{mr} \ A) \ \Leftrightarrow \ \forall y (x \ \mathsf{pmr}_y \ A)$

Embeddings IL into LL

$$
\begin{aligned}
(A \wedge B)^* &:\equiv A^* \,\&\, B^* \\
(A \vee B)^* &:\equiv\, !A^* \oplus !B^* \\
(A \rightarrow B)^* &:\equiv\, !A^* \multimap B^* \\
(\forall x A)^* &:\equiv \forall x A^* \\
(\exists x A)^* &:\equiv \exists x !A^*
\end{aligned}
$$

Embeddings IL into LL

$$
\begin{array}{llll}
(A \wedge B)^* & :\equiv A^* \,\&\, B^* & (A \wedge B)^\circ & :\equiv A^\circ \otimes B^\circ \\
(A \vee B)^* & :\equiv\, !A^* \oplus !B^* & (A \vee B)^\circ & :\equiv A^\circ \oplus B^\circ \\
(A \to B)^* & :\equiv\, !A^* \multimap B^* & (A \to B)^\circ & :\equiv\, !(A^\circ \multimap B^\circ) \\
(\forall x A)^* & :\equiv \forall x A^* & (\forall x A)^\circ & :\equiv\, !\forall x A^\circ \\
(\exists x A)^* & :\equiv \exists x !A^* & (\exists x A)^\circ & :\equiv \exists x A^\circ
\end{array}
$$

Embeddings IL into LL

$$
\begin{array}{llllll}
(A \wedge B)^* & :\equiv & A^* \,\&\, B^* & (A \wedge B)^\circ & :\equiv & A^\circ \otimes B^\circ \\
(A \vee B)^* & :\equiv & !A^* \oplus !B^* & (A \vee B)^\circ & :\equiv & A^\circ \oplus B^\circ \\
(A \to B)^* & :\equiv & !A^* \multimap B^* & (A \to B)^\circ & :\equiv & !(A^\circ \multimap B^\circ) \\
(\forall x A)^* & :\equiv & \forall x A^* & (\forall x A)^\circ & :\equiv & !\forall x A^\circ \\
(\exists x A)^* & :\equiv & \exists x !A^* & (\exists x A)^\circ & :\equiv & \exists x A^\circ
\end{array}
$$

Lemma

$A^\circ \multimapboth\, !A^*$

Realizability and LL

$$(A \to B)^* :\equiv\ !A^* \multimap B^*$$

$$(A \to B)^\circ :\equiv\ !(A^\circ \multimap B^\circ)$$

$$f \ \mathsf{pmr}_{x,w}\ A \to B :\equiv \forall y(x\ \mathsf{pmr}_y\ A) \to (fx\ \mathsf{pmr}_w\ B)$$

$$f\ \mathsf{mr}\ A \to B :\equiv \forall x((x\ \mathsf{mr}\ A) \to (fx\ \mathsf{mr}\ B))$$

Realizability and LL

$$(A \to B)^* \quad :\equiv \quad !A^* \multimap B^*$$

$$(A \to B)^\circ \quad :\equiv \quad !(A^\circ \multimap B^\circ)$$

$$f \ \mathsf{pmr}_{x,w} \ A \to B \quad :\equiv \quad \forall y (x \ \mathsf{pmr}_y \ A) \to (fx \ \mathsf{pmr}_w \ B)$$

$$f \ \mathsf{mr} \ A \to B \quad :\equiv \quad \forall x ((x \ \mathsf{mr} \ A) \to (fx \ \mathsf{mr} \ B))$$

Realizability and LL

$$(A \to B)^* \quad :\equiv \quad !A^* \multimap B^*$$

$$(A \to B)^\circ \quad :\equiv \quad !(A^\circ \multimap B^\circ)$$

$$f \,\mathsf{pmr}_{x,w}\, A \to B \quad :\equiv \quad \forall y (x \,\mathsf{pmr}_y\, A) \to (fx \,\mathsf{pmr}_w\, B)$$

$$f \,\mathsf{mr}\, A \to B \quad :\equiv \quad \forall x ((x \,\mathsf{mr}\, A) \to (fx \,\mathsf{mr}\, B))$$

Realizability and LL

$$(A \to B)^* :\equiv\ !A^* \multimap B^*$$
$$(A \to B)^\circ :\equiv\ !(A^\circ \multimap B^\circ)$$

$$f \ \mathsf{pmr}_{x,w}\ A \to B :\equiv \forall y(x \ \mathsf{pmr}_y\ A) \to (fx \ \mathsf{pmr}_w\ B)$$
$$f \ \mathsf{mr}\ A \to B :\equiv \forall x((x \ \mathsf{mr}\ A) \to (fx \ \mathsf{mr}\ B))$$

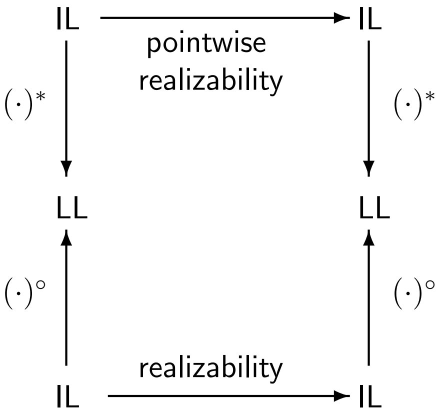
Lemma

$A^\circ \multimapboth\ !A^*$

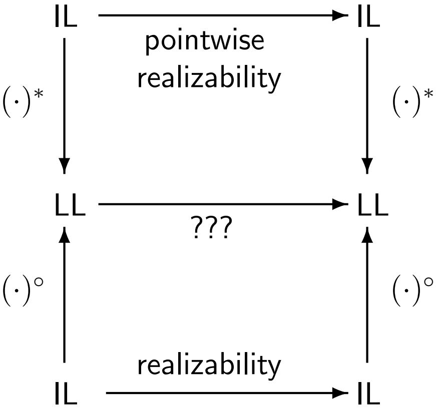
Lemma

$(x \ \mathsf{mr}\ A) \Leftrightarrow \forall y(x \ \mathsf{pmr}_y\ A)$

Realizability and LL



Realizability and LL



Outline

A model of LL

Interpret formulas A of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|^x_y)$
- two sets of nodes A^+, A^-
- edge relation $|A|^x_y$

A model of LL

Interpret formulas A of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|^x_y)$ (simultaneous game)
- two sets of nodes A^+, A^- (sets of moves)
- edge relation $|A|^x_y$ (adjudication relation)

A model of LL

Interpret formulas A of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|^x_y)$ (simultaneous game)
- two sets of nodes A^+, A^- (sets of moves)
- edge relation $|A|^x_y$ (adjudication relation)

$\mathcal{B}(X, Y)$ $\equiv$ bipartite graphs between X and Y
(set of possible games with move-sets X, Y)

A model of LL

Interpret formulas A of linear logic as **bipartite graphs**

- $(A^+, A^-, |A|^x_y)$ (simultaneous game)
- two sets of nodes A^+, A^- (sets of moves)
- edge relation $|A|^x_y$ (adjudication relation)

$\mathcal{B}(X, Y)$ $\equiv$ bipartite graphs between X and Y
(set of possible games with move-sets X, Y)

$\mathcal{B}_f(X, Y)$ $\equiv$ functional bipartite graphs between X and Y
(set of strategies in sequential version of game)

Some simple games

$$
\begin{aligned}
1 &:\equiv (\{*\}, \{*\}, \{\langle *, * \rangle\}) \\
\bot &:\equiv (\{*\}, \{*\}, \{\,\}) \\
0 &:\equiv (\{\,\}, \{*\}, \{\,\}) \\
\top &:\equiv (\{*\}, \{\,\}, \{\,\}).
\end{aligned}
$$

Dual of a game

Given bipartite graph $A \equiv (A^+, A^-, |A|)$ define

$$A^\perp \;:\equiv\; (A^-, A^+, \neg|A|).$$

Dual of a game

Given bipartite graph $A \equiv (A^+, A^-, |A|)$ define

$$A^\perp \;:\equiv\; (A^-, A^+, \neg|A|).$$

Lemma

- $A \sim (A^\perp)^\perp$
- $1 \sim \perp^\perp$
- $0 \sim \top^\perp$

where $\sim$ denotes graph isomorphism.

Sum of games

Play two games but only count outcome of one

$$|A \oplus B|^{\mathsf{inj}_i x}_{\langle y, w\rangle} :\equiv \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^x_w & \text{if } i = 1 \end{cases}$$

$$|A \mathbin{\&} B|^{\langle x, v\rangle}_{\mathsf{inj}_i y} :\equiv \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^v_y & \text{if } i = 1 \end{cases}$$

where $(A \oplus B)^+ = A^+ \uplus B^+$ and $(A \oplus B)^- = A^- \times B^-$.

Sum of games

Play two games but only count outcome of one

$$|A \oplus B|^{\mathsf{inj}_i x}_{\langle y, w \rangle} :\equiv \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^x_w & \text{if } i = 1 \end{cases}$$

$$|A \,\&\, B|^{\langle x, v \rangle}_{\mathsf{inj}_i y} :\equiv \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^v_y & \text{if } i = 1 \end{cases}$$

where $(A \oplus B)^+ = A^+ \uplus B^+$ and $(A \oplus B)^- = A^- \times B^-$.

Lemma

- $A \oplus 0 \sim A$
- $A \,\&\, \top \sim A$

Product of games

Play two games in parallel

$$
\begin{aligned}
|A \parr B|^{\langle S,T\rangle}_{\langle y,w\rangle} &\quad :\equiv \quad |A|^{Sw}_{y} \text{ or } |B|^{Ty}_{w} \\
|A \otimes B|^{\langle x,v\rangle}_{\langle S,T\rangle} &\quad :\equiv \quad |A|^{x}_{Sv} \text{ and } |B|^{v}_{Tx}
\end{aligned}
$$

where

- $(A \parr B)^{+} = \mathcal{B}_f(B^{-}, A^{+}) \times \mathcal{B}_f(A^{-}, B^{+})$
- $(A \parr B)^{-} = A^{-} \times B^{-}$.

Product of games

Play two games in parallel

$$|A \parr B|^{\langle S,T\rangle}_{\langle y,w\rangle} \quad :\equiv \quad |A|^{Sw}_{y} \text{ or } |B|^{Ty}_{w}$$

$$|A \otimes B|^{\langle x,v\rangle}_{\langle S,T\rangle} \quad :\equiv \quad |A|^{x}_{Sv} \text{ and } |B|^{v}_{Tx}$$

where

- $(A \parr B)^+ = \mathcal{B}_f(B^-, A^+) \times \mathcal{B}_f(A^-, B^+)$
- $(A \parr B)^- = A^- \times B^-$.

Lemma

- $A \parr \bot \sim A$
- $A \otimes 1 \sim A$

Relative games

Let $A \multimap B \;:\equiv\; A^{\perp} \mathbin{⅋} B$

In particular we have that

$$|A \multimap B|^{\langle S,T \rangle}_{\langle x,w \rangle} \;\equiv\; \text{if } |A|^{x}_{Sw} \text{ then } |B|^{Tx}_{w}$$

where

- $(A \multimap B)^+ = \mathcal{B}_f(A^+, B^+) \times \mathcal{B}_f(B^-, A^-)$
- $(A \multimap B)^- = A^+ \times B^-$.

Duplication of games

Play several copies of a game in parallel

$$|?A|^*_y \quad :\equiv \quad \exists x^{A^+} |A|^x_y$$
$$|!A|^x_* \quad :\equiv \quad \forall y^{A^-} |A|^x_y$$

where $(?A)^+ = \{*\}$ and $(?A)^- = A^-$.

Duplication of games

Play several copies of a game in parallel

$$|?A|^*_y \quad :\equiv \quad \exists x^{A^+} |A|^x_y$$
$$|!A|^x_* \quad :\equiv \quad \forall y^{A^-} |A|^x_y$$

where $(?A)^+ = \{*\}$ and $(?A)^- = A^-$.

Lemma

- $?0 \sim \bot$
- $!\top \sim 1$

Soundness

Theorem

If A is provable in linear logic then the bipartite graph A has a covering point, i.e. there exists an x^{A^+} such that $\forall y^{A^-} |A|^x_y$.

Soundness

Theorem

If A is provable in linear logic then the bipartite graph A has a covering point, i.e. there exists an x^{A^+} such that $\forall y^{A^-} |A|^x_y$.

A is provable $\Rightarrow$ first player has a winning move in game A

Intuitionistic truth via linear logic

Via $(\cdot)^\circ\colon$ IL $\mapsto$ LL we can model an intuitionistic formula A as the bipartite graph A°

More precisely, let $x \Vdash A \;\equiv\; \forall y^{(A^\circ)^-} |A^\circ|^x_y$

A intuitionistically true if $\exists x (x \Vdash A)$

Intuitionistic truth via linear logic

Via $(\cdot)^\circ$: IL $\mapsto$ LL we can model an intuitionistic formula A as the bipartite graph A°

More precisely, let $x \Vdash A \;\equiv\; \forall y^{(A^\circ)^-} |A^\circ|^x_y$

A intuitionistically true if $\exists x (x \Vdash A)$

Theorem

$$\begin{array}{lllll} \langle x, y \rangle & \Vdash A \wedge B & \Leftrightarrow & (x \Vdash A) \wedge (y \Vdash B) \\ \mathsf{inj}_i x & \Vdash A \vee B & \Leftrightarrow & (x \Vdash A) \lozenge_i (x \Vdash B) \\ S & \Vdash A \to B & \Leftrightarrow & \forall x((x \Vdash A) \to (Sx \Vdash B)). \end{array}$$

Outline

Functional interpretation of LL

Four changes from previous interpretation:

1. Work with infinite bipartite graphs
   X, Y sets of functionals of finite type
   (strategies $=$ functionals)

2. Define an interpretation of LL inside LL
   Adjudication relation as a formula of LL

3. Interpret quantifiers

4. Look at different interpretations of exponentials

Finite types

Assume a couple of basic types like $\mathbb{B}$ and $\mathbb{N}$

Close under

- Function type $\rho \to \tau$
- Product type $\rho \times \tau$
- List type ρ^*

Functional interpretation of LL

Additives

Play both games $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$

One of the players chooses which game will count

$$
\begin{aligned}
|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} \quad &:\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} \quad &:\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}
\end{aligned}
$$

where $A \Diamond_z B \equiv (!(z = \mathsf{tt}) \multimap A) \,\&\, (!(z = \mathsf{ff}) \multimap B)$.

Functional interpretation of LL

Quantifiers (Generalised additives)

Play all games $|A_z|^{\boldsymbol{x}}_{\boldsymbol{y}}$

One player chooses which game will count

Other player is allowed to know which game was chosen

$$|\exists z A_z|^{\boldsymbol{x},z}_{\boldsymbol{f}} \quad :\equiv \quad |A_z|^{\boldsymbol{x}}_{\boldsymbol{f}z}$$

$$|\forall z A_z|^{\boldsymbol{f}}_{\boldsymbol{y},z} \quad :\equiv \quad |A_z|^{\boldsymbol{f}z}_{\boldsymbol{y}}$$

Functional interpretation of LL

Multiplicatives

Play games $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ in parallel

One of the players can play copycat

$$
\begin{aligned}
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \parr |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}}
\end{aligned}
$$

Functional interpretation of LL

Exponentials (Generalised multiplicatives)

Play several copies of game $|A|_{\boldsymbol{y}}^{\boldsymbol{x}}$

One player must choose a uniform move

$$\begin{aligned} |?A|_{\boldsymbol{y}} \quad &:\equiv \quad ?\exists \boldsymbol{x} |A|_{\boldsymbol{y}}^{\boldsymbol{x}} \\ |!A|^{\boldsymbol{x}} \quad &:\equiv \quad !\forall \boldsymbol{y} |A|_{\boldsymbol{y}}^{\boldsymbol{x}} \end{aligned}$$

Other player plays second (break of symmetry)

Other player plays a set of moves

Functional interpretation of LL

Exponentials (Generalised multiplicatives)

Play several copies of game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

One player must choose a uniform move

$$
\begin{array}{lll}
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} \sqsubset \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|!A|^{\boldsymbol{x}}_{\boldsymbol{g}} & :\equiv & !\forall \boldsymbol{y} \sqsubset \boldsymbol{g}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{array}
$$

Other player plays second (break of symmetry)

Other player plays a set of moves

Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms η, ϵ and μ such that

(I) Every element x belongs to a set ηx

(II) The sets y_i are contained in the set $\epsilon y_0 y_1$

(III) For each $x \sqsubset b$ the set hx is contained in $\mu h b$

Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ such that

(I) Every element $\boldsymbol{x}$ belongs to a set $\boldsymbol{\eta x}$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\eta x}\, A \vdash A[\boldsymbol{x}/\boldsymbol{y}]$

(II) The sets $\boldsymbol{y}_i$ are contained in the set $\boldsymbol{\epsilon y}_0 \boldsymbol{y}_1$

(III) For each $\boldsymbol{x} \sqsubset \boldsymbol{b}$ the set $\boldsymbol{hx}$ is contained in $\boldsymbol{\mu h b}$

Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ such that

(I) Every element $\boldsymbol{x}$ belongs to a set $\boldsymbol{\eta x}$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\eta x}\, A \vdash A[\boldsymbol{x}/\boldsymbol{y}]$

(II) The sets $\boldsymbol{y}_i$ are contained in the set $\boldsymbol{\epsilon y}_0 \boldsymbol{y}_1$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\epsilon y}_0 \boldsymbol{y}_1\, A \vdash \forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i\, A \qquad (i \in \{0, 1\})$

(III) For each $\boldsymbol{x} \sqsubset \boldsymbol{b}$ the set $\boldsymbol{hx}$ is contained in $\boldsymbol{\mu h b}$

Exponentials: Conditions

The kind of move-sets need to satisfy:

There exists terms $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ such that

(I) Every element $\boldsymbol{x}$ belongs to a set $\boldsymbol{\eta x}$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\eta x}\, A \vdash A[\boldsymbol{x}/\boldsymbol{y}]$

(II) The sets $\boldsymbol{y}_i$ are contained in the set $\boldsymbol{\epsilon y}_0 \boldsymbol{y}_1$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\epsilon y}_0 \boldsymbol{y}_1\, A \vdash \forall \boldsymbol{y} \sqsubset \boldsymbol{y}_i\, A \qquad (i \in \{0, 1\})$

(III) For each $\boldsymbol{x} \sqsubset \boldsymbol{b}$ the set $\boldsymbol{hx}$ is contained in $\boldsymbol{\mu h b}$
$\forall \boldsymbol{y} \sqsubset \boldsymbol{\mu h b}\, A \vdash \forall \boldsymbol{x} \sqsubset \boldsymbol{b} \forall \boldsymbol{y} \sqsubset \boldsymbol{hx}\, A$.

Soundness

Theorem

Assuming (I, II, III). *If*

$$\mathsf{LL} \vdash A$$

there exists a closed simply typed λ*-term* t *such that*

$$\mathsf{LL}^{\omega} \vdash \forall y |A|^t_y.$$

Instances satisfying (I, II, III)

- **Whole set**

$$|!A|^{\boldsymbol{x}} :\equiv !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Instances satisfying (I, II, III)

- **Whole set**

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- **Finite sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

Instances satisfying (I, II, III)

- **Whole set**

  $|!A|^{\boldsymbol{x}} \; :\equiv \; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- **Finite sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \; :\equiv \; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- **Singleton sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \; :\equiv \; !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}.$

Instances satisfying (I, II, III)

- **Whole set**

  $|!A|^{\boldsymbol{x}} \; :\equiv \; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- **Finite sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \; :\equiv \; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

- **Singleton sets** (*assuming decidability*)

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \; :\equiv \; !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}.$

Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier

$$A \quad \mapsto \quad \exists\!\!\!/\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{y}}_{\boldsymbol{y}}$$

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a "double advantage"

Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier

$$A \quad \mapsto \quad \exists\!\!\!/\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{y}}_{\boldsymbol{y}}$$

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a "double advantage"
- Could we use sequential games?
- Can this "double advantage" be separated?

Functional interpretation of LL

- Symmetric game $\Rightarrow$ branching quantifier

$$A \quad \mapsto \quad \exists\!\!\!/\,\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{y}}_{\boldsymbol{y}}$$

- Characterisation principles more complicated
- Games $!A$ and $?A$ correspond to a "double advantage"
- Could we use sequential games?
- Can this "double advantage" be separated?

Yes, in intuitionistic linear logic

Outline

Simultaenous versus sequential games

Let us now work with **sequential games**

i.e. Eloise plays first, followed by Abelard's move

$$A \quad \mapsto \quad \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Simultaenous versus sequential games

Let us now work with **sequential games**

i.e. Eloise plays first, followed by Abelard's move

$$A \quad \mapsto \quad \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

No restriction, since Eloise's move could be a function

$$\exists \boldsymbol{f} \forall \boldsymbol{y} |A|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{y}} \quad \equiv \quad \forall \boldsymbol{y} \exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Functional interpretation of ILL

$$|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} \quad :\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} \quad :\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

Functional interpretation of ILL

$$
\begin{array}{lcl}
|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \lozenge_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \mathbin{\&} B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \lozenge_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A|^{\boldsymbol{x},z}_{\boldsymbol{y}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z A|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A|^{\boldsymbol{f}z}_{\boldsymbol{y}}
\end{array}
$$

Functional interpretation of ILL

$$
\begin{array}{lcl}
|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \lozenge_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \mathbin{\&} B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \lozenge_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A|^{\boldsymbol{x},z}_{\boldsymbol{y}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z A|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}}
\end{array}
$$

Functional interpretation of ILL

$$
\begin{array}{lcl}
|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \Diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A|^{\boldsymbol{x},z}_{\boldsymbol{y}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z A|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} & :\equiv & !\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{array}
$$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set** — Kreisel mod. realizability

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^{\circ}|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^{\circ}$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set**

  $|!A|^{\boldsymbol{x}} \; :\equiv \; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

  Kreisel mod. realizability

  $|A^{\circ}|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^{\circ}$

- **Finite sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \; :\equiv \; !\forall \boldsymbol{y} \in \boldsymbol{a} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set** — Kreisel mod. realizability

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^{\circ}|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^{\circ}$

- **Finite sets** — Diller-Nahm inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^{*}|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^{*}$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set** — Kreisel mod. realizability

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ $\qquad |A^{\circ}|^{\boldsymbol{x}} \multimap\!\!\circ\; (\boldsymbol{x} \text{ mr } A)^{\circ}$

- **Finite sets** — Diller-Nahm inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ $\qquad |A^{*}|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap\!\!\circ\; (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^{*}$

- **Singleton sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} \;:\equiv\; !|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

Instances satisfying (I, II, III)

Same three conditions need to be satisfied, and we have:

- **Whole set** — Kreisel mod. realizability

  $|!A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^\circ|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^\circ$

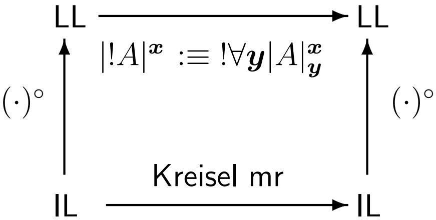
- **Finite sets** — Diller-Nahm inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^*$

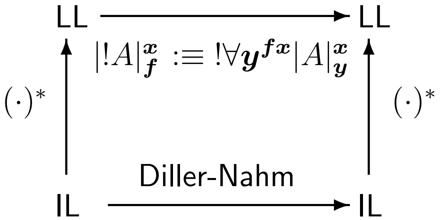
- **Singleton sets** — Gödel Dialectica inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} \;:\equiv\; !|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_D(\boldsymbol{x}; \boldsymbol{y}))^*$

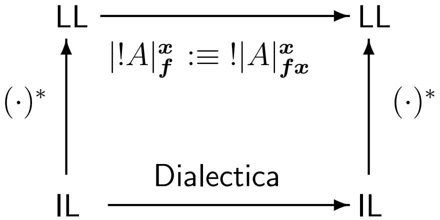
Realizability and LL



Realizability and LL



Realizability and LL



Question

Modified realizability interprets full **extensionality**

$$\forall x(fx = gx) \rightarrow Ff = Fg$$

Dialectica interprets **Markov principle**

$$\neg \forall x A_{\mathrm{qf}} \rightarrow \exists x \neg A_{\mathrm{qf}}$$

Can we combine both?

Question

Modified realizability interprets full **extensionality**

$$\forall x(fx = gx) \to Ff = Fg$$

Dialectica interprets **Markov principle**

$$\neg \forall x A_{\mathrm{qf}} \to \exists x \neg A_{\mathrm{qf}}$$

Can we combine both?

Yes (*thanks to the fact that* $!$ *is not cannonical*)

Multi-modal ILL

Add **three** different modalities $!_k A, !_d A$ and $!_g A$ with rules

$$\frac{!_X \Gamma \vdash A}{!_X \Gamma \vdash !_Y A} \, (!_r) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_Y A \vdash B} \, (!_l)$$

$$\frac{\Gamma, !_{Z_0} A, !_{Z_1} A \vdash B}{\Gamma, !_Y A \vdash B} \, (\mathrm{con}, \star) \qquad \frac{\Gamma \vdash B}{\Gamma, !_Y A \vdash B} \, (\mathrm{wkn})$$

where $X, Y, Z_i \in \{k > d > g\}$ and $X \geq Y \geq Z_i$

Multi-modal ILL

Add **three** different modalities $!_k A, !_d A$ and $!_g A$ with rules

$$\frac{!_X \Gamma \vdash A}{!_X \Gamma \vdash !_Y A}\ (!_r) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_Y A \vdash B}\ (!_l)$$

$$\frac{\Gamma, !_{Z_0} A, !_{Z_1} A \vdash B}{\Gamma, !_Y A \vdash B}\ (\mathrm{con}, \star) \qquad \frac{\Gamma \vdash B}{\Gamma, !_Y A \vdash B}\ (\mathrm{wkn})$$

where $X, Y, Z_i \in \{k > d > g\}$ and $X \geq Y \geq Z_i$

$(\star)$ Syntactic condition ensuring decidability when $Y = g$

Hybrid functional interpretation

Kreisel bang

$$|!_k A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Diller-Nahm bang

$$|!_d A|^{\boldsymbol{x}}_{\boldsymbol{f}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Gödel bang

$$|!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}} \;:\equiv\; !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}$$

Hybrid functional interpretation

Kreisel bang

$$|!_k A|^{\boldsymbol{x}} \;:\equiv\; !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Diller-Nahm bang

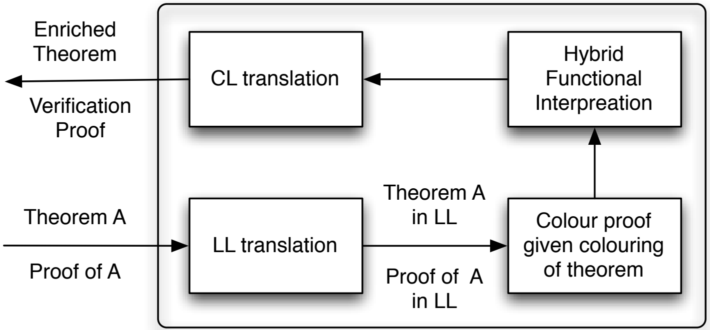
$$|!_d A|^{\boldsymbol{x}}_{\boldsymbol{f}} \;:\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

Gödel bang

$$|!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}} \;:\equiv\; !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}$$

Let a colouring algorithm decide the optimal/desired labelling

Hybrid functional interpretation



References

Modified realizability interpretation of classical linear logic
LICS 2007

Hybrid functional interpretations
with M.-D. Hernest, CiE 2008 (LNCS 5028:251-260, 2008)

Functional interpretations of linear and intuitionistic logic
To appear in I&C

Hybrid functional interpretations of linear and IL
To appear in JoL&C

Functional interpretations of intuitionistic linear logic
with G. Ferreira, in preparation