The Theory of Selection Functions

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23 November 2010

Outline

1. Quantifiers and Selection Functions
2. Finite and Infinite Products
3. Sequential Games

Outline

Quantifiers

$$\phi : (X \to R) \to R$$

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For instance:

Operation	ϕ	:	$(X \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Quantifiers

$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

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Quantifiers (Multi-valued)

$$\phi : (X \to R) \to 2^R \qquad (\equiv K_R X)$$

For instance:

Operation	ϕ	:	$(X \to R) \to 2^R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to 2^{\mathbb{B}}$
Supremum-i	$\sup^i_{[0,1]}$	:	$([0,1] \to \mathbb{R}^n) \to 2^{\mathbb{R}^n}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to 2^{\mathbb{R}}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to 2^{\bot}$
Fixed point operator	fix_X	:	$(X \to X) \to 2^X$

Theorem (Witness Theorem)

For any $p\colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$p(a) \quad \Leftrightarrow \quad \exists x^X p(x)$$

(similar to Hilbert's ε*-term)*

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(similar to Hilbert's ε*-term)*

Theorem (Counter-example Theorem)

For any $p\colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$p(a) \quad \Leftrightarrow \quad \forall x^X p(x)$$

(a *is counter-example to* p *if one exists)*

Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ there is a point $a \in [0,1]$ such that

$$p(a) = \int_0^1 p$$

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Theorem (Maximum Value Theorem)

For any $p \in [0,1] \to \mathbb{R}^n$ *there is a point* $a \in [0,1]$ *such that*

$$p(a) \in \sup{}^i \, p$$

Selection Functions

$$\varepsilon : (X \to R) \to X$$

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For instance:

Operation	ε	:	$(X \to R) \to X$
Hilbert's operator	ε	:	$(X \to \mathbb{B}) \to X$
Arg sup	$\mathrm{argsup}_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to [0,1]$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Attainable Quantifiers

Definition (Selection Functions for a Quantifier)

$\varepsilon \colon JX$ is called a **selection function** for $\phi \colon KX$ if

$$p(\varepsilon p) \in \phi(p)$$

holds for all $p \colon X \to R$

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Definition (Attainable Quantifiers)

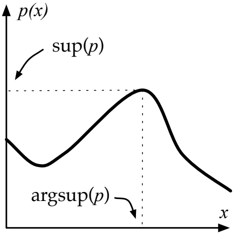
A quantifier $\phi \colon KX$ is called **attainable** if it has a selection function $\varepsilon \colon JX$

Attainable Quantifiers: Examples

- $\sup\colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier

$$\sup(p) = p(\operatorname{argsup}(p))$$

where $\operatorname{argsup}\colon J_{\mathbb{R}}[0,1]$.



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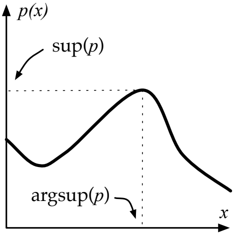
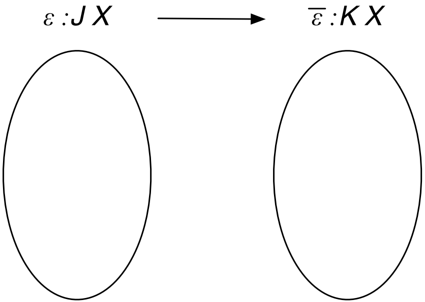


- $\mathsf{fix}\colon K_X X$ is an attainable quantifier

  $$\mathsf{fix}(p) = p(\mathsf{fix}(p))$$

  where $\mathsf{fix}\colon J_X X \; (= K_X X)$.

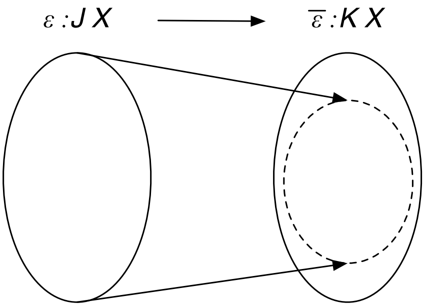
From Selection Functions to Quantifiers



Every selection function $\varepsilon \colon JX$ defines a quantifier $\overline{\varepsilon} \colon KX$

$$\overline{\varepsilon}(p) \quad = \quad p(\varepsilon(p))$$

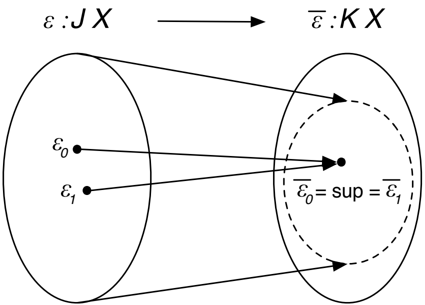
From Selection Functions to Quantifiers



Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$$\phi(p) = 0$$

From Selection Functions to Quantifiers



Different ε might define same ϕ, e.g. $X = [0,1]$ and $R = \mathbb{R}$

$$\varepsilon_0(p) = \mu x . \sup p = p(x)$$
$$\varepsilon_1(p) = \nu x . \sup p = p(x)$$

Outline

Nested quantifiers $\equiv$ single quantifier on **product space**

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$$\sup_x \int_0^1 p(x,y)dy \quad \stackrel{\mathbb{R}}{\equiv} \quad (\sup \otimes \textstyle\int)(p^{[0,1]^2 \to \mathbb{R}})$$

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Definition (Product of Single-valued Quantifiers)

Given $\phi \colon KX$ and $\psi \colon KY$ define $\phi \otimes \psi \ : \ K(X \times Y)$

$$(\phi \otimes \psi)(p) \stackrel{R}{:\equiv} \phi(\lambda x^X.\psi(\lambda y^Y.p(x,y)))$$

where $p \colon X \times Y \to R$.

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$$(\phi \otimes \psi)(p) :\stackrel{R}{\equiv} \phi(\lambda x^X . \psi(\lambda y^Y . p(x,y)))$$

where $p \colon X \times Y \to R$.

Does not work with multi-valued quantifiers!

Quantifier Elimination

Suppose X and Y are such that for some ε and δ

$$\begin{aligned} \exists x^X \, p(x) &= p(\varepsilon p) \\ \forall y^Y \, p(y) &= p(\delta p). \end{aligned}$$

Quantifier Elimination

Suppose X and Y are such that for some ε and δ

$$\begin{aligned} \exists x^X \, p(x) &= p(\varepsilon p) \\ \forall y^Y \, p(y) &= p(\delta p). \end{aligned}$$

Then

$$\exists x^X \forall y^Y \, p(x, y) = \exists x \, p(x, b(x))$$

where

$$b(x) = \delta(\lambda y. p(x, y))$$

Quantifier Elimination

Suppose X and Y are such that for some ε and δ

$$\begin{aligned} \exists x^X\, p(x) &= p(\varepsilon p) \\ \forall y^Y\, p(y) &= p(\delta p). \end{aligned}$$

Then

$$\begin{aligned} \exists x^X \forall y^Y\, p(x,y) &= \exists x\, p(x, b(x)) \\ &= p(a, b(a)) \end{aligned}$$

where

$$\begin{aligned} b(x) &= \delta(\lambda y. p(x,y)) \\ a &= \varepsilon(\lambda x. p(x, b(x))). \end{aligned}$$

Product of Selection Functions

Definition (Product of Selection Functions)

Given $\varepsilon \colon JX$ and $\delta \colon JY$ define $\varepsilon \otimes \delta \colon J(X \times Y)$ as

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{:=} (a, b(a))$$

where

$$\begin{array}{lcl} b(x) & = & \delta(\lambda y.p(x,y)) \\ a & = & \varepsilon(\lambda x.p(x,b(x))). \end{array}$$

Homomorphism Lemma

Lemma

$$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$$

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Lemma

$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$

Proof.

$(\overline{\varepsilon \otimes \delta})(q) = q(a, b_a) = \overline{\varepsilon}(\lambda x.q(x, b_x)) = \overline{\varepsilon}(\lambda x.\overline{\delta}(\lambda y.q(x, y))) = (\overline{\varepsilon} \otimes \overline{\delta})(q).$ □

Homomorphism Lemma

Lemma

$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$

Proof.

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Corollary

If $\phi \colon KX$ and $\psi \colon KY$ are attainable single-valued quantifiers with sel. fct. $\varepsilon \colon JX$ and $\delta \colon JY$ then

$$\overline{\varepsilon \otimes \delta} = \phi \otimes \psi$$

Definition (Iterated Product – Finite)

Given $\varepsilon_i \colon JX_i$, $0 \leq i \leq n$, define $(\bigotimes_{i=k}^{n} \varepsilon_i) \colon J\Pi_{i=k}^{n} X_i$ as

$$\left(\bigotimes_{i=k}^{n} \varepsilon_i \right) = \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{n} \varepsilon_i \right)$$

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Definition (Iterated Product – Infinite)

Given $\varepsilon_i \colon JX_i$, $i \in \mathbb{N}$, define $(\bigotimes_{i \geq k} \varepsilon_i) \colon J\Pi_{i \geq k} X_i$ as

$$\left(\bigotimes_{i \geq k} \varepsilon_i\right) = \varepsilon_k \otimes \left(\bigotimes_{i \geq k+1} \varepsilon_i\right)$$

for $q \colon \Pi_i X_i \to R$ continuous and $R = \mathbb{N}$ (assumed henceforth)

Theorem (Idempotency)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$, *let*

$$\alpha \overset{\Pi_{i \geq 0} X_i}{=} \left(\bigotimes_{i \geq 0} \varepsilon_i \right) (q)$$

then, for all k,

$$\mathsf{tail}^k(\alpha) \overset{\Pi_{i \geq k} X_i}{=} \left(\bigotimes_{i \geq k} \varepsilon_i \right) (q_{[\alpha](k)})$$

Theorem (Idempotency)

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Proof.

By course-of-values induction on k □

Theorem (Product Quantifier)

Given attainable $\phi_i \colon KX_i$, *with sel. func.* $\varepsilon_i \colon JX_i$, *and* $q \colon \Pi_i X_i \to R$, *there exist* $p_i \colon X_i \to R$ *such that*

$$q(\alpha) \; = \; \left(\overline{\bigotimes_{i \geq 0} \varepsilon_i} \right) (q) \; \in \; \bigcap_i \phi_i(p_i)$$

(α *as before)*

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(α as before)

Proof.

Take $p_i = \lambda y_i . (\overline{\bigotimes_{k \geq i} \varepsilon_k})(q_{[\alpha](i) * y_i})$

Recall that $p_i(\varepsilon_i(p_i)) \in \phi_i(p_i)$

Then $p_i(\varepsilon_i(p_i)) = p_i(\alpha(i)) = q(\alpha)$ (Idempotency theorem) □

Corollary (Spector Equation)

Given attainable quantifiers $\phi_i \colon KX_i$, *with selection functions* $\varepsilon_i \colon JX_i$, *and* $q \colon \Pi X_i \to R$, *there exist* α *and* p_i *such that*

$$\begin{aligned} \alpha(i) &= \varepsilon_i(p_i) \\ q(\alpha) &\in \phi_i(p_i) \qquad (\textit{for all } i) \end{aligned}$$

Corollary (Spector Equation)

Given attainable quantifiers $\phi_i \colon KX_i$*, with selection functions* $\varepsilon_i \colon JX_i$*, and* $q \colon \Pi X_i \to R$*, there exist* α *and* p_i *such that*

$$\begin{array}{rcll} \alpha(i) & = & \varepsilon_i(p_i) & \\ q(\alpha) & \in & \phi_i(p_i) & \quad (\text{for all } i) \end{array}$$

Proof.

Take α and p_i as before, i.e.

$p_i = \lambda y_i.(\overline{\bigotimes_{k \geq i} \varepsilon_k})(q_{[\alpha](i) * y_i})$

$\alpha = (\bigotimes_{k \geq i} \varepsilon_k)(q)$ □

Theorem (Optimal Strategy)

Given attainable $\phi_i \colon KX_i$, *with sel. func.* $\varepsilon_i \colon JX_i$, *and* $q \colon \Pi_i X_i \to R$, *there exist* $\alpha_k \colon \Pi_{i<k} X_i \to X_k$ *such that*

$$q(\alpha^{\vec{x}}) \;\in\; \phi_k(\lambda y_k . q(\alpha^{\vec{x}, y_k})) \qquad (\vec{x} = x_0, \ldots, x_{k-1})$$

where $\alpha^{\vec{x}}(i) = x_i$ *if* $i < k$ *and* $\alpha_i([\alpha^{\vec{x}}](i))$ *otherwise*

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where $\alpha^{\vec{x}}(i) = x_i$ *if* $i < k$ *and* $\alpha_i([\alpha^{\vec{x}}](i))$ *otherwise*

Proof.

Take $\alpha_k(\vec{x}) = \pi_0((\bigotimes_{k \geq i} \varepsilon_i)(q_{\vec{x},y_i}))$

We have $\alpha^{\vec{x}} = (\bigotimes_{k \geq i} \varepsilon_i)(q_{\vec{x}})$ (Idempotency thm)

Use Product Quantifier theorem □

Outline

Sequential Games

Definition

A Game is a tuple $(R, (X_i)_{i \in \mathbb{N}}, (\phi_i)_{i \in \mathbb{N}}, q)$ where

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon K_R X_i$ is the **goal (mul.-val.) quantifier** for round i
- $q \colon \Pi_{i \in \mathbb{N}} X_i \to R$ is the **outcome function**

with q determined after **finitely** many moves

Definition (Strategy)

Family of mappings $\mathrm{next}_k \colon \prod_{i=0}^{k-1} X_i \to X_k$

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Family of mappings $\text{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$

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$$b^{\vec{a}}_i = \mathsf{next}_i(\vec{a}, b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{i-1})$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) \in \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))$$

Product of selection functions computes optimal strategies!

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Corollary

For any game with attainable goal quantifiers $\phi_i \colon KX_i$ *an optimal strategy can be computed as*

$$\mathsf{next}_k(\vec{x}) = \pi_0 \left(\left(\bigotimes_{i \geq k} \varepsilon_i \right) (q_{\vec{x}}) \right)$$

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$$\mathsf{next}_k(\vec{x}) = \pi_0\left(\left(\bigotimes_{i \geq k} \varepsilon_i\right)(q_{\vec{x}})\right)$$

Proof.

Follows directly from Optimal Strategy theorem □

Standard Game Theory

When $R = \mathbb{R}^n$ and ϕ_i are $\max^i$ or $\sup^i$

(attainable quantifiers with selection functions argsup^i)

Generalised Game	$\mapsto$	Standard Game
Optimal strategy	$\mapsto$	Strategy in Nash equilibrium
Product of argsup^i	$\mapsto$	Backward induction!

Proof Theory

Computational interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x, r) \quad \mapsto \quad \forall \varepsilon_{(\cdot)} \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

Proof Theory

Computational interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x, r) \quad \mapsto \quad \underline{\forall \varepsilon_{(.)}} \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

ε's define quantifiers, which partially define a game

Proof Theory

Computational interpretation

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ε's define quantifiers, which partially define a game

Computational interpretation relies on completing the definition of the game so optimal strategy solves problem

Open Questions

1. Relation of product of selection functions $\otimes$ to the different product BBC (over system T)

$$\mathsf{BBC}(\varepsilon)(q) = \lambda n.\varepsilon_n \left(\lambda x_n.\overline{\mathsf{BBC}(\varepsilon)}(q_{(n,x_n)})\right)$$

(due to Berardi, Bezem, Coquand)

Open Questions

1. Relation of product of selection functions $\bigotimes$ to the different product BBC (over system T)

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2. Does BBC compute optimal strategies in some different (but also natural) notion of game

References

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