The Theory of Selection Functions

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Outline

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Quantifiers

$$\phi : (X \to R) \to R$$

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For instance:

Operation	ϕ	:	$(X \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Quantifiers

$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

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Quantifiers (Multi-valued)

$$\phi : (X \to R) \to 2^R \qquad (\equiv K_R X)$$

For instance:

Operation	ϕ	:	$(X \to R) \to 2^R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to 2^{\mathbb{B}}$
Supremum-i	$\sup^i_{[0,1]}$	:	$([0,1] \to \mathbb{R}^n) \to 2^{\mathbb{R}^n}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to 2^{\mathbb{R}}$
Double negation	$\neg\neg X$	:	$(X \to \perp) \to 2^{\perp}$
Fixed point operator	fix_X	:	$(X \to X) \to 2^X$

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Theorem (Witness Theorem)

For any $p \colon X \to \mathbb{B}$ there is a point $a \in X$ such that

$$p(a) \quad \Leftrightarrow \quad \exists x^X p(x)$$

(similar to Hilbert's ε-term)

Theorem (Witness Theorem)

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(similar to Hilbert's ε-term)

Theorem (Counter-example Theorem)

For any $p \colon X \to \mathbb{B}$ there is a point $a \in X$ such that

$$p(a) \quad \Leftrightarrow \quad \forall x^X p(x)$$

(a is counter-example to p if one exists)

Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$p(a) = \int_0^1 p$$

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Theorem (Maximum Value Theorem)

For any $p \in [0,1] \to \mathbb{R}^n$ *there is a point* $a \in [0,1]$ *such that*

$$p(a) \in \sup{}^i\, p$$

Selection Functions

$$\varepsilon \; : \; (X \to R) \to X$$

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For instance:

Operation			
Hilbert's operator	ε	:	$(X \to \mathbb{B}) \to X$
Arg sup	$\mathrm{argsup}_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to [0,1]$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Attainable Quantifiers

Definition (Selection Functions for a Quantifier)

$\varepsilon\colon JX$ is called a **selection function** for $\phi\colon KX$ if

$$p(\varepsilon p) \in \phi(p)$$

holds for all $p\colon X \to R$

Attainable Quantifiers

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Definition (Attainable Quantifiers)

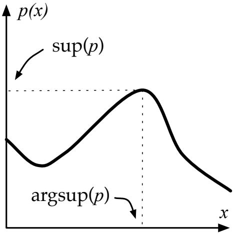
A quantifier $\phi \colon KX$ is called **attainable** if it has a selection function $\varepsilon \colon JX$

Attainable Quantifiers: Examples

- $\sup\colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier

$$p(\mathrm{argsup}(p)) = \sup(p)$$

where $\mathrm{argsup}\colon J_{\mathbb{R}}[0,1]$.



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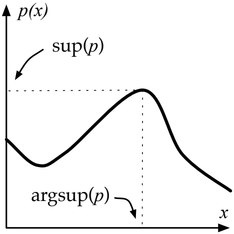
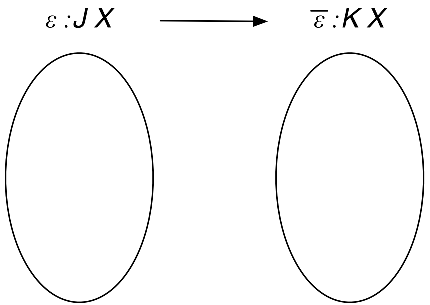


- fix: $K_X X$ is an attainable quantifier

$$p(\mathsf{fix}(p)) \in \mathsf{fix}(p)$$

where fix: $J_X X \; (= K_X X)$.

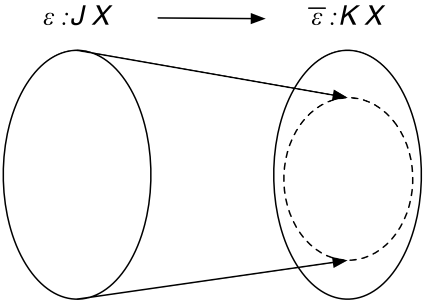
From Selection Functions to Quantifiers



Every selection function $\varepsilon \colon JX$ defines a quantifier $\overline{\varepsilon} \colon KX$

$$\overline{\varepsilon}(p) \quad = \quad p(\varepsilon(p))$$

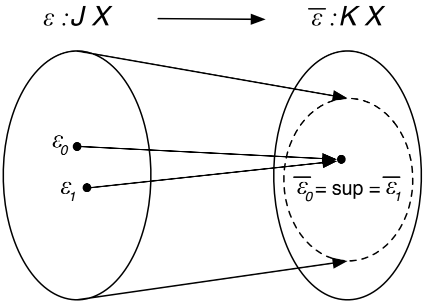
From Selection Functions to Quantifiers



Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$$\phi(p) \quad = \quad 0$$

From Selection Functions to Quantifiers



Different ε might define same ϕ, e.g. $X = [0,1]$ and $R = \mathbb{R}$

$$\begin{aligned} \varepsilon_0(p) &= \mu x . \sup p = p(x) \\ \varepsilon_1(p) &= \nu x . \sup p = p(x) \end{aligned}$$

Outline

Describing “goal”

Q: How much would you like to pay for your flight?

Describing "goal"

Q: How much would you like to pay for your flight?

A: As little as possible!

Quantifiers: Game Theoretic Reading

R = set of outcomes

X = set of possible moves

$$\boxed{\phi \in (X \to R) \to R}$$

describes the desired outcome $\phi p \in R$ given $p \in X \to R$

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R = set of outcomes

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describes the desired outcome $\phi p \in R$ given $p \in X \to R$

In the example:

$$\begin{array}{lcl} R & = & \textit{prices (real numbers)} \\ X & = & \textit{possible flights} \\ X \to R & = & \textit{price of each flight} \\ \phi & = & \textit{minimal value functional} \end{array}$$

Sequential Games

Definition

A Game is a tuple $(R, (X_i)_{i\in\mathbb{N}}, (\phi_i)_{i\in\mathbb{N}}, q)$ where

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon K_R X_i$ is the **goal (mul.-val.) quantifier** for round i
- $q \colon \Pi_{i\in\mathbb{N}} X_i \to R$ is the **outcome function**

with q determined after **finitely** many moves

Definition (Strategy)

Family of mappings $\mathsf{next}_k \colon \prod_{i=0}^{k-1} X_i \to X_k$

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Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, b_{k+1}^{\vec{a}}, \ldots$ where

$$b_i^{\vec{a}} = \mathsf{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$

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$$b^{\vec{a}}_i = \mathsf{next}_i(\vec{a}, b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{i-1})$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) \in \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))$$

Standard Game Theory

When $R = \mathbb{R}^n$ and ϕ_i are $\max^i$ or $\sup^i$

(attainable quantifiers with selection functions argsup^i)

Generalised Game	$\mapsto$	Standard Game
Optimal strategy	$\mapsto$	Strategy in Nash equilibrium

Outline

Nested quantifiers $\equiv$ single quantifier on **product space**

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Nested quantifiers $\equiv$ single quantifier on **product space**

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Definition (Product of Single-valued Quantifiers)

Given $\phi \colon KX$ and $\psi \colon KY$ define $\phi \otimes \psi \,:\, K(X \times Y)$

$$(\phi \otimes \psi)(p) :\stackrel{R}{\equiv} \phi(\lambda x^X . \psi(\lambda y^Y . p(x,y)))$$

where $p \colon X \times Y \to R$.

Nested quantifiers $\equiv$ single quantifier on **product space**

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$$(\phi \otimes \psi)(p) :\stackrel{R}{\equiv} \phi(\lambda x^X.\psi(\lambda y^Y.p(x,y)))$$

where $p \colon X \times Y \to R$.

Does not work with multi-valued quantifiers!

Quantifier Elimination

Suppose X and Y are such that for some ε and δ

$$\begin{aligned} \exists x^X \, p(x) &= p(\varepsilon p) \\ \forall y^Y \, p(y) &= p(\delta p). \end{aligned}$$

Quantifier Elimination

Suppose X and Y are such that for some ε and δ

$$\begin{aligned} \exists x^X\, p(x) &= p(\varepsilon p) \\ \forall y^Y\, p(y) &= p(\delta p). \end{aligned}$$

Then

$$\exists x^X \forall y^Y\; p(x, y) = \exists x\; p(x, b(x))$$

where

$$b(x) = \delta(\lambda y . p(x, y))$$

Quantifier Elimination

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Then

$$\begin{aligned} \exists x^X \forall y^Y \, p(x, y) &= \exists x \, p(x, b(x)) \\ &= p(a, b(a)) \end{aligned}$$

where

$$\begin{aligned} b(x) &= \delta(\lambda y . p(x, y)) \\ a &= \varepsilon(\lambda x . p(x, b(x))). \end{aligned}$$

Product of Selection Functions

Definition (Product of Selection Functions)

Given $\varepsilon \colon JX$ and $\delta \colon JY$ define $\varepsilon \otimes \delta \colon J(X \times Y)$ as

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{:=} (a, b(a))$$

where

$$\begin{array}{lcl} b(x) & = & \delta(\lambda y.p(x,y)) \\ a & = & \varepsilon(\lambda x.p(x, b(x))). \end{array}$$

Homomorphism Lemma

Lemma

$$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$$

Homomorphism Lemma

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$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$

Proof.

$(\overline{\varepsilon \otimes \delta})(q) = q(a, b_a) = \overline{\varepsilon}(\lambda x.q(x, b_x)) = \overline{\varepsilon}(\lambda x.\overline{\delta}(\lambda y.q(x, y))) = (\overline{\varepsilon} \otimes \overline{\delta})(q).$ □

Definition (Iterated Product – Finite)

Given $\varepsilon_i \colon JX_i$, $0 \leq i \leq n$, define $(\bigotimes_{i=k}^{n} \varepsilon_i) \colon J\Pi_{i=k}^{n} X_i$ as

$$\left(\bigotimes_{i=k}^{n} \varepsilon_i \right) = \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{n} \varepsilon_i \right)$$

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Definition (Iterated Product – Infinite)

Given $\varepsilon_i \colon JX_i$, $i \in \mathbb{N}$, define $(\bigotimes_{i \geq k} \varepsilon_i) \colon J\Pi_{i \geq k} X_i$ as

$$\left(\bigotimes_{i \geq k} \varepsilon_i\right) = \varepsilon_k \otimes \left(\bigotimes_{i \geq k+1} \varepsilon_i\right)$$

for $q \colon \Pi_i X_i \to R$ continuous and $R = \mathbb{N}$ (assumed henceforth)

Product of Quantifiers

Theorem

The infinite product of quantifiers does not exist in $\mathcal{C}$ (the model of continuous functionals) even assuming R discrete.

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Proof.

Let $\phi_i = \exists_{X_i}$. We have that

$$\left(\bigotimes_{i \geq 0} \exists_{X_i}\right)(\mathrm{true})$$

is true iff all X_i are non-empty. But continuity implies only finitely many X_i are checked. $\square$

Lemma (Unfolding)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$ *we have*

$$\left(\bigotimes_{i \geq 0} \varepsilon_i\right)(q) \stackrel{\Pi_i X_i}{=} a_0 * \left(\bigotimes_{i \geq 1} \varepsilon_i\right)(q_{a_0})$$

where

$$a_0 = \varepsilon_0\left(\lambda x_0 . q_{x_0}\left(\left(\left(\bigotimes_{i \geq 1} \varepsilon_i\right)(q_{x_0})\right)\right)\right)$$

Lemma (Unfolding)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$ *we have*

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where

$$a_0 = \varepsilon_0\left(\lambda x_0 . q_{x_0}\left(\left(\bigotimes_{i \geq 1} \varepsilon_i\right)(q_{x_0})\right)\right)$$

Proof.

Unfolding definition of $\otimes$ □

Lemma (Iterated Unfolding)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$, *let*

$$\alpha \stackrel{\Pi_{i \geq 0} X_i}{=} \left(\bigotimes_{i \geq 0} \varepsilon_i \right) (q)$$

then, for all k,

$$\alpha(k) \stackrel{X_k}{=} \varepsilon_k(\lambda x^{X_k}. \left(\overline{\bigotimes_{i \geq k+1} \varepsilon_i} \right) (q_{[\alpha](k) * x}))$$

Lemma (Iterated Unfolding)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$*, let*

$$\alpha \stackrel{\Pi_{i\geq 0} X_i}{=\!=} \left(\bigotimes_{i\geq 0} \varepsilon_i \right) (q)$$

then, for all k*,*

$$\alpha(k) \stackrel{X_k}{=\!=} \varepsilon_k(\lambda x^{X_k} . \left(\overline{\bigotimes_{i\geq k+1} \varepsilon_i} \right) (q_{[\alpha](k) * x}))$$

Proof.

Induction + Unfolding Lemma □

Theorem (Idempotency)

Given $\varepsilon_i \colon JX_i$ *and* $q \colon \Pi_i X_i \to R$, *let*

$$\alpha \overset{\Pi_{i \geq 0} X_i}{=\!=} \left(\bigotimes_{i \geq 0} \varepsilon_i \right) (q)$$

then, for all k,

$$\mathsf{tail}^k(\alpha) \overset{\Pi_{i \geq k} X_i}{=\!=} \left(\bigotimes_{i \geq k} \varepsilon_i \right) (q_{[\alpha](k)})$$

Theorem (Idempotency)

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Proof.

By the Iterated Unfolding Lemma □

Theorem (Product Quantifier)

Given attainable $\phi_i \colon KX_i$*, with sel. func.* $\varepsilon_i \colon JX_i$*, and* $q \colon \Pi_i X_i \to R$*, there exist* $p_i \colon X_i \to R$ *such that*

$$q(\alpha) \;=\; \left(\overline{\bigotimes_{i\geq 0} \varepsilon_i}\right)(q) \;\in\; \bigcap_i \phi_i(p_i)$$

(α *as before)*

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(α *as before)*

Proof.

Take $p_i = \lambda y_i . (\overline{\bigotimes_{k \geq i} \varepsilon_k})(q_{[\alpha](i) * y_i})$

Recall that $p_i(\varepsilon_i(p_i)) \in \phi_i(p_i)$

Then $p_i(\varepsilon_i(p_i)) = p_i(\alpha(i)) = q(\alpha)$ (Idempotency Thm) □

Corollary (Spector Equation – Variant)

Given attainable quantifiers $\phi_i \colon KX_i$, with selection functions $\varepsilon_i \colon JX_i$, and $q \colon \Pi X_i \to R$, there exist α and p_i such that

$$\begin{array}{rcll} \alpha(i) & = & \varepsilon_i(p_i) & \\ q(\alpha) & \in & \phi_i(p_i) & \qquad (\textit{for all } i) \end{array}$$

Corollary (Spector Equation – Variant)

Given attainable quantifiers $\phi_i \colon KX_i$*, with selection functions* $\varepsilon_i \colon JX_i$*, and* $q \colon \Pi X_i \to R$*, there exist* α *and* p_i *such that*

$$\begin{array}{rcll} \alpha(i) & = & \varepsilon_i(p_i) & \\ q(\alpha) & \in & \phi_i(p_i) & \qquad (\textit{for all } i) \end{array}$$

Proof.

Take α and p_i as before, i.e.

$p_i = \lambda y_i.(\overline{\bigotimes_{k \geq i} \varepsilon_k})(q_{[\alpha](i)*y_i})$

$\alpha = (\bigotimes_{i \geq 0} \varepsilon_i)(q)$ □

Theorem (Optimal Strategy)

Given attainable $\phi_i : KX_i$ *and* $q : \Pi_i X_i \to R$, *there exist* $\mathsf{next}_k : \Pi_{i<k} X_i \to X_k$ *such that*

$$q(\mathbf{b}^{\vec{x}}) \;\in\; \phi_k(\lambda y_k . q(\mathbf{b}^{\vec{x}, y_k})) \qquad (\vec{x} = x_0, \ldots, x_{k-1})$$

where $\mathbf{b}^{\vec{x}}(i) = x_i$ *if* $i < k$ *and* $\mathsf{next}_i(\vec{x}, b^{\vec{x}}_k, \ldots, b^{\vec{x}}_{i-1})$ *otherwise*

Theorem (Optimal Strategy)

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where $\mathbf{b}^{\vec{x}}(i) = x_i$ *if* $i < k$ *and* $\mathsf{next}_i(\vec{x}, b^{\vec{x}}_k, \ldots, b^{\vec{x}}_{i-1})$ *otherwise*

Proof.

Take $\mathsf{next}_k(\vec{x}) = \pi_0((\bigotimes_{i \geq k} \varepsilon_i)(q_{\vec{x}}))$

We have $\mathbf{b}^{\vec{x}} = (\bigotimes_{i \geq k} \varepsilon_i)(q_{\vec{x}})$ (Idempotency thm)

Use Product Quantifier theorem □

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