Sequential Games and Optimal Strategies

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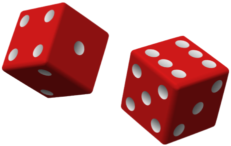

LIAMF, USP

São Paulo, 20 October 2011

Outline

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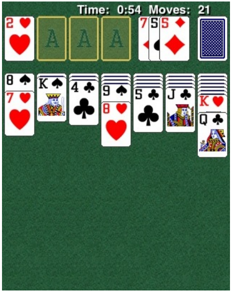
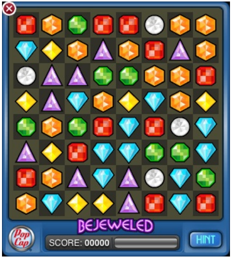
Single-player Games





Two-player Games

Two **players**: Black and White

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Possible **outcomes**:

- Black wins
- White wins
- Draw

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Strategy: Choice of move at round k given previous moves

Another Game

Two **players**: John and Julia

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John splits a cake. Julia chooses one of the two pieces

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Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

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Two **players**: John and Julia

John splits a cake. Julia chooses one of the two pieces

Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

Best strategy for John is to split cake into half

It is not a "winning strategy" but it is an **optimal strategy**

It maximises his payoff

Traditional Game Theory

Game defined via:

- Set of **players** P
- Sets of **moves** X_i for each player $i \in P$
- Set of **outcomes** R
- **Preference relations** on R for each player $i \in P$
- **Outcome function** mapping plays to outcomes

Set of Players vs Number of Rounds

Number of players is not essential

It is important what the “goal” at each round is

Rounds with “**same goal**” mean played by “**same player**”

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How to describe the goal at a particular round?

You could say: The goal is to win!

But maybe this is not possible (or might not even make sense)

Instead, the goal should be described as

a choice of outcome from each set of possible outcomes

As in...

Q: How much would you like to pay for your flight?

As in...

Q: How much would you like to pay for your flight?

A: As little as possible!

Target function

If $R =$ set of outcomes and $X =$ set of possible moves then

$$\phi \in (X \to R) \to R$$

describes the desired outcome $\phi p \in R$ given that the outcome of the game $px \in R$ for each move $x \in X$ is given.

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In the example:

X	$=$	*possible flights*
R	$=$	*real number*
$X \to R$	$=$	*price of each flight*
ϕ	$=$	*minimal value functional*

Outline

Generalised quantifiers

$$\phi : (X \to R) \to R$$

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For instance

Operation	ϕ	:	$(X \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Limit	$\lim$	:	$(\mathbb{N} \to R) \to R$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Generalised quantifiers

$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

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Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\int_0^1 p = p(a)$$

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Theorem (Maximum Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\sup p = p(a)$$

Theorem (Witness Theorem)

For any $p \colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$\exists x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(similar to Hilbert's ε*-term).*

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Theorem (Counter-example Theorem)

For any $p \colon X \to \mathbb{B}$ there is a point $a \in X$ such that

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(a is counter-example to p if one exists).

Let $JX \equiv (X \to R) \to X$

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Definition (Selection Functions)

$\varepsilon \colon JX$ is called a **selection function** for $\phi \colon (X \to R) \to R$ if

$$p(\varepsilon p) = \phi(p)$$

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Definition (Attainable Quantifiers)

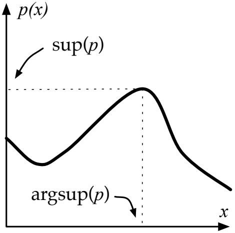
A generalised quantifier $\phi \colon KX$ is called **attainable** if it has a selection function $\varepsilon \colon JX$

For Instance

- $\sup\colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier as

$$\sup(p) = p(\operatorname{argsup}(p))$$

where $\operatorname{argsup}\colon J_{\mathbb{R}}[0,1]$



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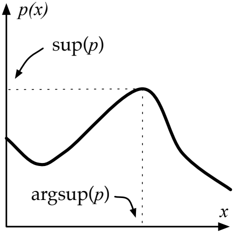
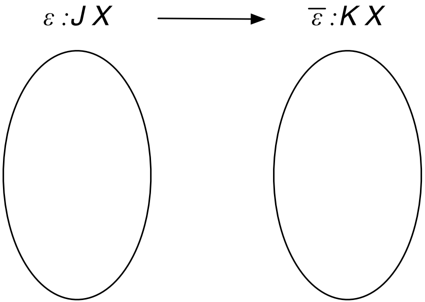


- $\mathsf{fix}\colon K_X X$ is an attainable quantifier as
$$\mathsf{fix}(p) = p(\mathsf{fix}(p))$$
where $\mathsf{fix}\colon J_X X \; (= K_X X)$

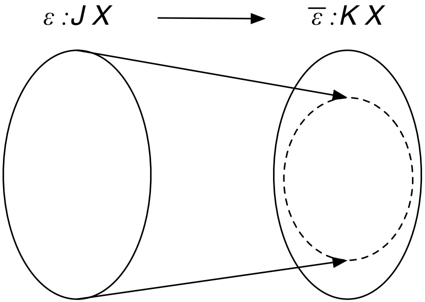
Selection Functions and Generalised Quantifiers



Every selection function $\varepsilon\colon JX$ defines a quantifier $\overline{\varepsilon}\colon KX$

$$\overline{\varepsilon}(p) \quad = \quad p(\varepsilon(p))$$

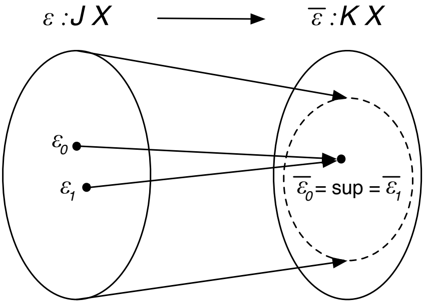
Selection Functions and Generalised Quantifiers



Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$$\phi(p) \;=\; 0$$

Selection Functions and Generalised Quantifiers



Different ε might define same ϕ, e.g. $X = [0,1]$ and $R = \mathbb{R}$

$$\begin{aligned} \varepsilon_0(p) &= \mu x . \sup p = p(x) \\ \varepsilon_1(p) &= \nu x . \sup p = p(x) \end{aligned}$$

Outline

Finite Sequential Games

Definition (A tuple $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$ where)

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon (X_i \to R) \to 2^R$ is the **goal quantifier** for round i
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Definition (Strategy)

Family of mappings

$$\text{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$$

Optimal Strategies

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$

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Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) \in \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))$$

Examples

Example (Nash Equilibrium with common payoff)

Moves X_i	Sets of moves
Outcomes R	Payoff $\mathbb{R}$
Goal quantifier ϕ_i	Maximal value function
Outcome function q	Payoff function $q\colon \Pi_{i=0}^{n-1} X_i \to \mathbb{R}$

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Optimal strategy

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) \quad = \quad \mathrm{argsup}_{x_k} \sup_{x_{k+1}} \ldots \sup_{x_{n-1}} q(\vec{x})$$

Examples

Example (Satisfiability)

Moves X_i	Booleans $\mathbb{B}$
Outcomes R	Boolean $\mathbb{B}$
Goal quantifier ϕ_i	Existential quantifier $\exists\colon K_{\mathbb{B}}\mathbb{B}$
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THE ROYAL SOCIETY

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Optimal strategy

$$\text{next}_k(x_0, \ldots, x_{k-1}) \quad = \quad x_k \text{ such that } \exists x_{k+1} \ldots \exists x_{n-1} q(\vec{x}) \text{ (if possible)}$$

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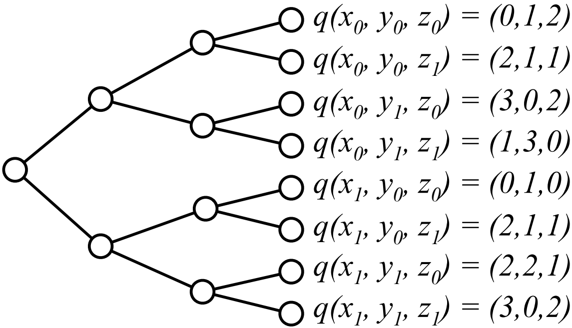
Backward Induction (Classical Game Theory)

Three players, payoff function $q\colon X \times Y \times Z \to \mathbb{R}^3$

Each player is trying to maximise their own payoff



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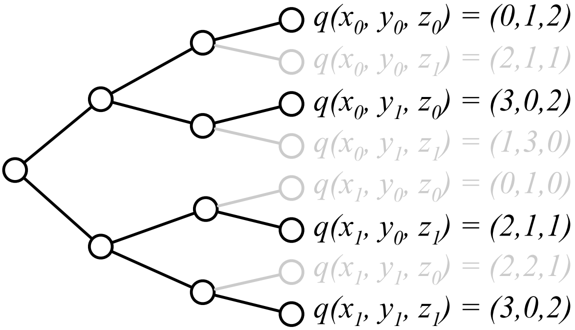

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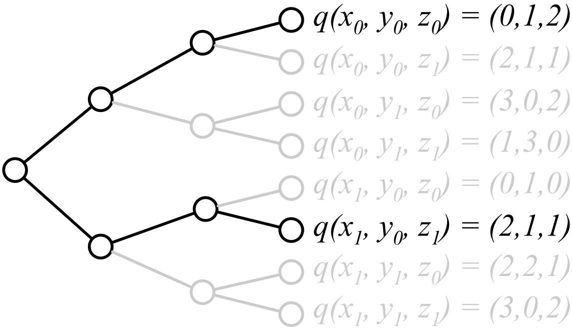

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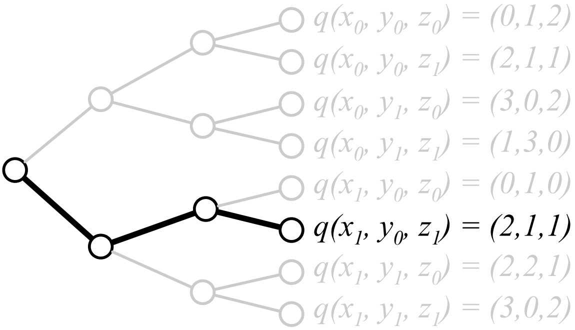

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Let $\operatorname{argmax}_i \colon (X_i \to \mathbb{R}^n) \to X_i$ find a point $x \in X_i$ at which the function $p \colon X_i \to \mathbb{R}^n$ has maximal i-value

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Consider n player. Given $q \colon \Pi_{i=0}^{n-1} X_i \to \mathbb{R}^n$, define

$$\mathsf{BI}(s) \overset{\Pi_{j=|s|}^{n-1} X_j}{=} \begin{cases} [\,] & \text{if } n = |s| \\ c_s * \mathsf{BI}(s * c_i) & \text{otherwise} \end{cases}$$

where $c_s = \mathrm{argmax}_{|s|}(\lambda x . q(s * x * \mathsf{BI}(s * x)))$

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where $c_s = \mathrm{argmax}_{|s|}(\lambda x.q(s * x * \mathsf{BI}(s * x)))$

Each player's **optimal strategy** can be described as

$$\mathsf{next}_i(s) = \mathrm{argmax}_{|s|}(\underbrace{\lambda x.q(s * x * \mathsf{BI}(s * x))}_{p \colon X_{|s|} \to \mathbb{R}^n})$$

Spector's Bar Recursion (1962)

Let

$$s\colon X^* \qquad q\colon X^* \to R \qquad \varepsilon_s\colon J_R X$$

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Given s, ω and ε_s define

$$\mathsf{BR}(s) \stackrel{X^*}{=} \begin{cases} [\,] & \text{if } \ n = |s| \\ c * \mathsf{BR}(s * c) & \text{otherwise} \end{cases}$$

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where $c = \varepsilon_s(\lambda x.q(s * x * \mathsf{BR}(s * x)))$

Spector actually defined a much more general recursion scheme where stopping condition depends on the play s

Main Theorem

Theorem (Escardó/O.'2011)

Given game (R, X_i, ϕ_i, q), if ϕ_i are attainable with selection functions ε_i then

$$\mathsf{next}(s) \stackrel{X}{=} (\mathsf{BR}(s))_0$$

is an **optimal strategy**, *i.e.*

$$q(s * \mathbf{b}^s) \in \phi_{|s|}(\lambda x.q(s * x * \mathbf{b}^{s*x}))$$

where $\mathbf{b}^s$ is the strategic extension of partial play s

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  $KA \to A$ corresponds to **double negation elimination**

  $JA \to A$ corresponds to **Peirce's law**
- Calculation of strategies in general corresponds to Spector's bar recursion, used in the proof of **consistency of classical analysis**

References

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