Nash Equilibrium
Bekič's Lemma
Backtracking and
Bar Recursion

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(based on jww Martín Escardó)

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Outline

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Sequential Payoff Games

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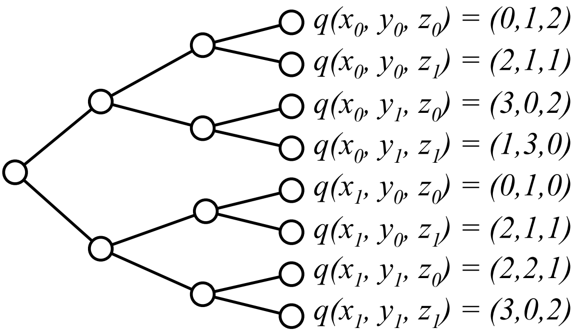
- **strategy profile** is a tuple $(\mathsf{next}_i)_{1 \leq i \leq n}$
- A strategy profile is in (Nash) **equilibrium** if no single player has an incentive to unilaterally change his strategy

Backward Induction

Three players, payoff function $q \colon X \times Y \times Z \to \mathbb{R}^3$

Each player is trying to maximise their own payoff



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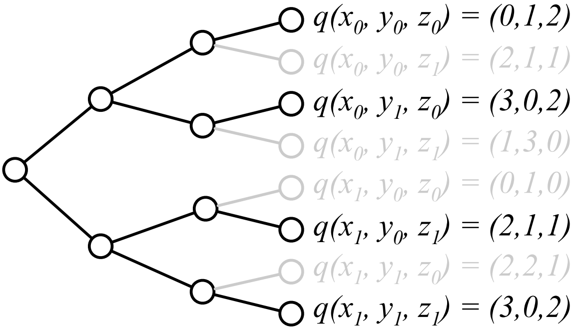

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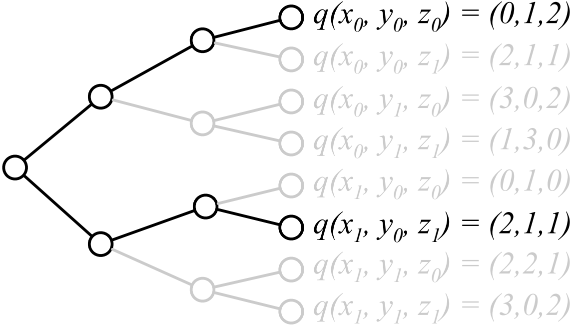

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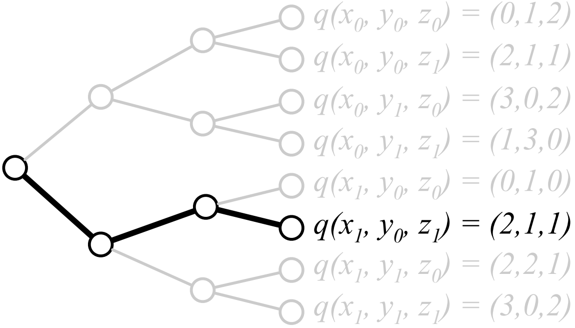

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BI: $\Pi_{j \leq i} X_j \to \Pi_{j > i} X_j$

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find $x \in X_i$ where $p \colon X_i \to \mathbb{R}^n$ has maximal i-value

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compute BI(s) assuming we have BI$(s * x)$ for all x

fix payoff function $q \colon \Pi_{i=1}^n X_i \to \mathbb{R}^n$

$$\mathsf{BI}(s) \overset{\Pi_{j > |s|} X_j}{=} \begin{cases} [\,] & \text{if } |s| = n \\ c_s * \mathsf{BI}(s * c_s) & \text{if } |s| < n \end{cases}$$

where $c_s = \mathrm{argmax}_{|s|+1}(\lambda x. q(s * x * \mathsf{BI}(s * x)))$

Equilibrium Strategy Profile

Let

$$\mathsf{BI}(s) \overset{\Pi_{j=|s|+1}^{n} X_j}{=} \begin{cases} [\,] & \text{if } |s| = n \\ c_s * \mathsf{BI}(s * c_s) & \text{if } |s| < n \end{cases}$$

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where $c_s = \operatorname{argmax}_{|s|+1}(\lambda x.q(s * x * \mathsf{BI}(s * x)))$

Each player's **optimal strategy** can be described as

$$\mathsf{next}_i(s) = \operatorname{argmax}_i(\underbrace{\lambda x.q(s * x * \mathsf{BI}(s * x))}_{p\colon X_i \to \mathbb{R}^n})$$

Outline

Bekič's Lemma

A mapping $\mathsf{fix}\colon (X \to X) \to X$ is a **fixed point operator** if

$$p(\mathsf{fix}\ p) = \mathsf{fix}\ p$$

for all $p\colon X \to X$

Bekič's Lemma

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Theorem

If each space X_i has a fixed point operator

$$\text{fix}_i \colon (X_i \to X_i) \to X_i$$

then so does the product space $X_1 \times \ldots \times X_n$

Bekič's Lemma – Construction

$\mathsf{BL}\colon \Pi_{j\leq i} X_j \to \Pi_{j>i} X_j$

fixed point over $\Pi_{j>i} X_j$ assuming $s\colon \Pi_{j\leq i} X_j$ fixed

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$\widetilde{\mathsf{fix}}_i\colon (X_i \to \Pi_{j=1}^{n} X_j) \to X_i$

find an i-fixed point of mappings $X_i \to \Pi_{j=1}^{n} X_j$

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compute $\mathsf{BL}(s)$ assuming we have $\mathsf{BL}(s * x)$ for all x

given $q\colon \Pi_{i=1}^n X_i \to \Pi_{i=1}^n X_i$

$$\mathsf{BL}(s) \overset{\Pi_{j>|s|} X_j}{=} \begin{cases} [\,] & \text{if } |s| = n \\ c_s * \mathsf{BL}(s * c_s) & \text{if } |s| < n \end{cases}$$

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where $c_s = \mathsf{fix}_{|s|+1}(\lambda x. q(s * x * \mathsf{BL}(s * x)))$

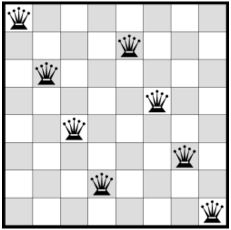
Hence, a fixed point of q is

$$\mathsf{BL}([\,]) = [x_1, \ldots, x_n]$$

Outline

The Problem

Place eight queens on chess board so none capture the other



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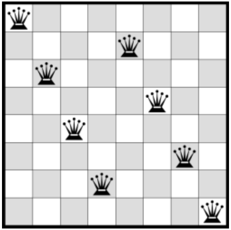

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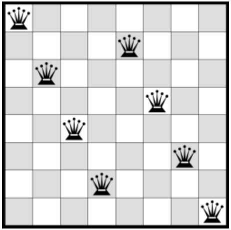

$\mathbf{8} = \{1, 2, \ldots, 8\}$

$q \colon \mathbf{8}^8 \to \mathbb{B}$ checks whether solution is correct

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given $q \colon \mathbf{8}^8 \to \mathbb{B}$ (*checking correctness of proposed solution*)

$$\mathsf{EQ}(s) \overset{\mathbf{8}^{8-|s|}}{=} \begin{cases} [\,] & \text{if } |s| = n \\ c_s * \mathsf{EQ}(s * c_s) & \text{if } |s| < n \end{cases}$$

where $c_s = \varepsilon(\lambda x.q(s * x * \mathsf{EQ}(s * x)))$

Eight Queens - Solution

Let

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A solution to the problem is given as

$$\mathsf{EQ}([\,]) = [x_1, \ldots, x_n]$$

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Interpreting Finite Choice

Finite Choice

$$\forall i \leq n \exists x \forall r A_i(x, r) \to \exists s \forall i \leq n \forall r A_i(s_i, r)$$

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Given $\varepsilon_i \colon (X \to R) \to X$ *such that*

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and $q \colon X^n \to R$ *produce* $s \colon X^n$ *such that*

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compute $\mathsf{BR}(s)$ assuming we have $\mathsf{BR}(s * x)$ for all x

given "counter-example function" $q\colon X^n \to R$

$$\mathsf{BR}(s) \overset{X^{|s|-n}}{=} \begin{cases} [\,] & \text{if } |s| = n \\ c_s * \mathsf{BR}(s * c_s) & \text{if } |s| < n \end{cases}$$

where $c_s = \varepsilon_{|s|+1}(\lambda x. q(s * x * \mathsf{BR}(s * x)))$

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Take

$$s = \text{BR}([\,])$$

Outline

Spector's Bar Recursion

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$$s\colon X^* \qquad \omega\colon X^{\mathbb{N}} \to \mathbb{N} \qquad q\colon X^* \to R \qquad \varepsilon_s\colon J_R X$$

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$$\mathsf{BR}_s(\omega)(\varepsilon)(q) \overset{X^*}{=} \begin{cases} [\,] & \text{if } |s| > \omega(\hat{s}) \\ c * \mathsf{BR}_{s*c}(\omega)(\varepsilon)(q) & \text{otherwise} \end{cases}$$

where $c = \varepsilon_s(\lambda x.q(s * x * \mathsf{BR}_{s*x}(\omega)(\varepsilon)(q)))$

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$$\mathsf{EPS}_s(\omega)(\varepsilon)(q) \stackrel{X^*}{=} \begin{cases} [\,] & \text{if } |s| > \omega(\hat{s}) \\ c * \mathsf{EPS}_{s*c}(\omega)(\varepsilon)(q) & \text{otherwise} \end{cases}$$

where $c = \varepsilon_s(\lambda x.q(s * x * \mathsf{EPS}_{s*x}(\omega)(\varepsilon)(q)))$

This is actually the **iterated product of selection functions**

T-equivalent to Spector's restricted form of bar recursion

Product of Selection Functions

Given $\otimes\colon J_R X \times (X \to J_R Y) \to J_R(X \times Y)$

Controlled product of selection functions

$$\mathsf{EPS}_s(\omega)(\varepsilon) \overset{J_R X^*}{=} \begin{cases} \lambda q.[\,] & \text{if } |s| > \omega(\hat{s}) \\ \varepsilon_s \otimes (\lambda x.\mathsf{EPS}_{s*x}(\omega)(\varepsilon)) & \text{otherwise} \end{cases}$$

Product of Selection Functions

EPS gives **direct** realisers as

- $\lambda\varepsilon, q, n.\mathsf{EPS}_{[\,]}(n)(\varepsilon)(q)$ realises

  FC : $\forall n(\forall i \leq n \exists x A_i(x) \rightarrow \exists s \forall i \leq n A_i(s_i))$

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- $\lambda\varepsilon, q, \omega.\mathsf{EPS}_{[\,]}(\omega)(\tilde{\varepsilon})(q)$ realises $(\tilde{\varepsilon}_s = \varepsilon_{|s|})$

  AC$_0$: $\forall n \exists x A_n(x) \to \exists \alpha \forall n A_n(\alpha(n))$

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- $\lambda\varepsilon, q, \omega.\mathsf{EPS}_{[\,]}(\omega)(\varepsilon)(q)$ realises

  DC : $\forall s \exists x A_s(x) \to \exists \alpha \forall n A_{\overline{\alpha}n}(\alpha(n))$

Further Information

- M. Escardó and P. Oliva
  Selection functions, bar recursion and backward induction
  MSCS, 20(2):127-168, 2010
- M. Escardó and P. Oliva
  Sequential games and optimal strategies
  Proceedings of the Royal Society A, 2011
- P. Oliva and T. Powell
  On Spector's bar recursion
  Final draft available