Some Connections Between Proof Theory and Game Theory

Paulo Oliva

Queen Mary University of London

CHoCoLa Meeting

Lyon, 10 May 2012

Outline

1. Brief Overview
   - Hintikka games (Classical Logic)
   - Lorenzen games (Intuitionistic Logic)
   - Blass games (Linear Logic)
2. Functional Interpretations
   - Strategies as moves
   - Realizability and dialectica
3. Quantifiers and Selection Functions
   - von Neumann games
   - A generalization
   - Interpreting countable and dependent choice

Outline

1. Brief Overview
   - Hintikka games (Classical Logic)
   - Lorenzen games (Intuitionistic Logic)
   - Blass games (Linear Logic)
2. Functional Interpretations
   - Strategies as moves
   - Realizability and dialectica
3. Quantifiers and Selection Functions
   - von Neumann games
   - A generalization
   - Interpreting countable and dependent choice

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Given games A_0 and A_1 one defines the game

- $A_0 \vee A_1$: verifier picks $i \in \{0, 1\}$, continue playing A_i

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Given games A_0 and A_1 one defines the game

- $A_0 \vee A_1$: verifier picks $i \in \{0, 1\}$, continue playing A_i
- $A_0 \wedge A_1$: falsifier picks $i \in \{0, 1\}$, continue playing A_i

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Given games A_0 and A_1 one defines the game

- $A_0 \vee A_1$: verifier picks $i \in \{0, 1\}$, continue playing A_i
- $A_0 \wedge A_1$: falsifier picks $i \in \{0, 1\}$, continue playing A_i
- Same for $\exists x A$ and $\forall x A$, except players choose $a \in M$

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Given games A_0 and A_1 one defines the game

- $A_0 \vee A_1$: verifier picks $i \in \{0, 1\}$, continue playing A_i
- $A_0 \wedge A_1$: falsifier picks $i \in \{0, 1\}$, continue playing A_i
- Same for $\exists x A$ and $\forall x A$, except players choose $a \in M$
- $\neg A$: swap roles, and continue playing A

Hintikka Games

Fix a model M of a first-order language

Two players **P** and **O**

Initial roles: **P** is the verifier, **O** is the falsifier

For atomic formula Q, verifier wins if Q holds in M

Given games A_0 and A_1 one defines the game

- $A_0 \vee A_1$: verifier picks $i \in \{0, 1\}$, continue playing A_i
- $A_0 \wedge A_1$: falsifier picks $i \in \{0, 1\}$, continue playing A_i
- Same for $\exists x A$ and $\forall x A$, except players choose $a \in M$
- $\neg A$: swap roles, and continue playing A

Theorem (Hintikka and Kulas, 1983)

$M \models A$ iff **P** *has a winning strategy in game A (over M)*

Lorenzen Games

- Lorenzen (1961)

- Two players $\{\mathbf{P}, \mathbf{O}\}$ debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond

Lorenzen Games

- Lorenzen (1961)

- Two players $\{\mathbf{P}, \mathbf{O}\}$ debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond

- Motivation: alternative semantics for IL
  If formula is provable in IL then **P** has winning strategy

Lorenzen Games

- Lorenzen (1961)

- Two players {**P**, **O**} debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond

- Motivation: alternative semantics for IL
  If formula is provable in IL then **P** has winning strategy
- Felscher (1985) found conditions for completeness
  Formula is provable in IL iff **P** has winning strategy

Lorenzen Games – Particle Rules

Ways a formula can be attacked/defended

Depends on the main connective/quantifier

Lorenzen Games – Particle Rules

Ways a formula can be attacked/defended

Depends on the main connective/quantifier

Let $X, Y \in \{$**P**, **O**$\}$ with $X \neq Y$, and $i < j < k$

Conjunction

(i)	X asserts	$A_1 \wedge A_2$
(j)	Y attacks (i) asserting	$\wedge_1$ (or $\wedge_2$)
(k)	X responds (j) asserting	A_1 (or A_2)

Lorenzen Games – Particle Rules

Ways a formula can be attacked/defended

Depends on the main connective/quantifier

Let $X, Y \in \{\mathbf{P}, \mathbf{O}\}$ with $X \neq Y$, and $i < j < k$

Conjunction

(i)	X	asserts	$A_1 \wedge A_2$
(j)	Y	attacks (i) asserting	$\wedge_1$ (or $\wedge_2$)
(k)	X	responds (j) asserting	A_1 (or A_2)

Disjunction

(i)	X	asserts	$A_1 \vee A_2$
(j)	Y	attacks (i) asserting	$\vee$
(k)	X	responds (j) asserting	A_1 (or A_2)

Lorenzen Games – Particle Rules

Let $X, Y \in \{\mathbf{P}, \mathbf{O}\}$ with $X \neq Y$, and $i < j < k$

Implication

(i)	X	asserts	$A \to B$
(j)	Y	attacks (i) asserting	A
(k)	X	responds (j) asserting	B

Lorenzen Games – Particle Rules

Let $X, Y \in \{\mathbf{P}, \mathbf{O}\}$ with $X \neq Y$, and $i < j < k$

Implication

(i)	X	asserts	$A \to B$
(j)	Y	attacks (i) asserting	A
(k)	X	responds (j) asserting	B

Negation

(i)	X	asserts	$\neg A$
(j)	Y	attacks (i) asserting	A
(k)	X	has no possible response to (j)	

Lorenzen Games – E.g. $P \wedge Q \rightarrow Q \wedge P$

Possible play in this game:

(0) **P** starts by asserting $P \wedge Q \rightarrow Q \wedge P$

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$

Lorenzen Games – E.g. $P \land Q \to Q \land P$

Possible play in this game:

(0)	**P** starts by asserting	$P \land Q \to Q \land P$
(1)	**O** attacks (0) asserting	$P \land Q$
(2)	**P** attacks (1) asserting	$\land_1$
(3)	**O** responds (2) asserting	P

Lorenzen Games – E.g. $P \wedge Q \rightarrow Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \rightarrow Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$
(3)	**O** responds (2) asserting	P
(4)	**P** attacks (1) asserting	$\wedge_2$

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$
(3)	**O** responds (2) asserting	P
(4)	**P** attacks (1) asserting	$\wedge_2$
(5)	**O** responds (4) asserting	Q

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$
(3)	**O** responds (2) asserting	P
(4)	**P** attacks (1) asserting	$\wedge_2$
(5)	**O** responds (4) asserting	Q
(6)	**P** responds (1) asserting	$Q \wedge P$

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$
(3)	**O** responds (2) asserting	P
(4)	**P** attacks (1) asserting	$\wedge_2$
(5)	**O** responds (4) asserting	Q
(6)	**P** responds (1) asserting	$Q \wedge P$
(7)	**O** attacks (6) asserting	$\wedge_1$

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0)	**P** starts by asserting	$P \wedge Q \to Q \wedge P$
(1)	**O** attacks (0) asserting	$P \wedge Q$
(2)	**P** attacks (1) asserting	$\wedge_1$
(3)	**O** responds (2) asserting	P
(4)	**P** attacks (1) asserting	$\wedge_2$
(5)	**O** responds (4) asserting	Q
(6)	**P** responds (1) asserting	$Q \wedge P$
(7)	**O** attacks (6) asserting	$\wedge_1$
(8)	**P** responds (7) asserting	Q

Lorenzen Games – Structural Rules

General organisation of the game:

S1 **P** may only assert atomic formulas already asserted by **O**

Lorenzen Games – Structural Rules

General organisation of the game:

- S1 **P** may only assert atomic formulas already asserted by **O**
- S2 A player can only respond the latest open attack

Lorenzen Games – Structural Rules

General organisation of the game:

- S1 **P** may only assert atomic formulas already asserted by **O**
- S2 A player can only respond the latest open attack
- S3 An attack may be responded at most once

Lorenzen Games – Structural Rules

General organisation of the game:

- S1 **P** may only assert atomic formulas already asserted by **O**
- S2 A player can only respond the latest open attack
- S3 An attack may be responded at most once
- S4 A **P**-assertion may be attacked at most once

Lorenzen Games – Structural Rules

General organisation of the game:

- S1 **P** may only assert atomic formulas already asserted by **O**
- S2 A player can only respond the latest open attack
- S3 An attack may be responded at most once
- S4 A **P**-assertion may be attacked at most once
- S5 **O** can only attack/respond the preceding **P**-assertion

Lorenzen Games – Structural Rules

General organisation of the game:

S1 **P** may only assert atomic formulas already asserted by **O**

S2 A player can only respond the latest open attack

S3 An attack may be responded at most once

S4 A **P**-assertion may be attacked at most once

S5 **O** can only attack/respond the preceding **P**-assertion

Remark: Dropping S2 and S3 gives semantics for CL!

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$\dfrac{\dfrac{\overline{\qquad\qquad\qquad}}{\vdots}}{\dfrac{\overline{\qquad\qquad\qquad}}{\textbf{P}\text{ asserts } P \wedge Q \to Q \wedge P}}$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree
P *chooses path from below, directed by* **O***-attacks*
O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$\dfrac{\dfrac{\dfrac{\textbf{O}\text{ asserts } P \wedge Q}{\vdots}}{}}{\textbf{P}\text{ asserts } P \wedge Q \to Q \wedge P}\ (\textbf{O}\text{ attacks with } \to)$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$\dfrac{\dfrac{\dfrac{\textbf{O}\text{ asserts } P \wedge Q}{\vdots}\,(\textbf{P}\text{ attacks with } \wedge_2, \wedge_1)}{}}{\textbf{P}\text{ asserts } P \wedge Q \to Q \wedge P}\,(\textbf{O}\text{ attacks with } \to)$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$\dfrac{\dfrac{\dfrac{\mathbf{O}\text{ asserts }P \wedge Q}{\mathbf{O}\text{ asserts }Q, P}\ (\mathbf{P}\text{ attacks with }\wedge_2, \wedge_1)}{\vdots}}{\mathbf{P}\text{ asserts }P \wedge Q \to Q \wedge P}\ (\mathbf{O}\text{ attacks with }\to)$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$\dfrac{\textbf{O}\text{ asserts } P \wedge Q}{\textbf{O}\text{ asserts } Q, P}\ (\textbf{P}\text{ attacks with } \wedge_2, \wedge_1)$$

$$\vdots$$

$$\dfrac{\overline{\textbf{P}\text{ asserts } Q \wedge P}}{\textbf{P}\text{ asserts } P \wedge Q \to Q \wedge P}\ (\textbf{O}\text{ attacks with } \to)$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O**-*attacks*

O *chooses path from above, directed by* **P**-*attacks*

For instance, play in example above corresponds to:

$$\frac{\textbf{O} \text{ asserts } P \wedge Q}{\textbf{O} \text{ asserts } Q, P}\ (\textbf{P} \text{ attacks with } \wedge_2, \wedge_1)$$

$$\vdots$$

$$\cfrac{\cfrac{}{\textbf{P} \text{ asserts } Q \wedge P}\ (\textbf{O} \text{ attacks with } \wedge_1)}{\textbf{P} \text{ asserts } P \wedge Q \to Q \wedge P}\ (\textbf{O} \text{ attacks with } \to)$$

Lorenzen Games – Intuition

A play is a path in a possible proof tree

P *chooses path from below, directed by* **O***-attacks*

O *chooses path from above, directed by* **P***-attacks*

For instance, play in example above corresponds to:

$$
\dfrac{\dfrac{\dfrac{\textbf{O}\text{ asserts } P \wedge Q}{\textbf{O}\text{ asserts } Q, P}\ (\textbf{P}\text{ attacks with } \wedge_2, \wedge_1)}{\vdots} \qquad \dfrac{\textbf{P}\text{ asserts } Q}{\textbf{P}\text{ asserts } Q \wedge P}\ (\textbf{O}\text{ attacks with } \wedge_1)}{\textbf{P}\text{ asserts } P \wedge Q \to Q \wedge P}\ (\textbf{O}\text{ attacks with } \to)
$$

Blass Games

Blass'1992

Games for **affine logic** (linear logic plus weakening)

Based on operations on infinite games devised in 1972

Blass Games

Blass'1992

Games for **affine logic** (linear logic plus weakening)

Based on operations on infinite games devised in 1972

Two main differences to Lorenzen games:

- Infinitely long plays (means not all games are determined)
- Two kinds of connectives (only one re-attackable)

Blass Games

Blass'1992

Games for **affine logic** (linear logic plus weakening)

Based on operations on infinite games devised in 1972

Two main differences to Lorenzen games:

- Infinitely long plays (means not all games are determined)
- Two kinds of connectives (only one re-attackable)

Can dispense with structural rule!

Blass Games – Definition

Two players **P** and **O**

A **Blass game** consists of an ordered triple (M, p, G) where

- M is the set of possible moves at each round
- $p \in \{$**P**, **O**$\}$ is the starting player
  (from then on players move alternatively)
- $G \subseteq M^\omega$ is the set of plays won by **P**

Game Operations – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$

Game Operations – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$

The new game $G_0 \,\&\, G_1$ is defined as

- **O** starts and chooses $i \in \{0, 1\}$
- Game G_i is then played

Game Operations – Conjunctions

Given games $G_0 = (M_0, s_0, G_0)$ and $G_1 = (M_1, s_1, G_1)$

The new game $G_0 \,\&\, G_1$ is defined as

- **O** starts and chooses $i \in \{0, 1\}$
- Game G_i is then played

The new game $G_0 \otimes G_1$ is defined as

- both games are played intertwined
- **O** plays when its his turn in both sub-games
  He chooses one of the games and makes a move there
- **P** plays when he is to move in either G_0 or G_1
- **O** wins if he wins in one of the sub-games

Blass Games

- The dual of a game is simply a swapping of roles
- Disjunctions follow by de Morgan
- Given game interpretation of atomics $P \mapsto G_P$ extend to game interpretation G_A for all formulas

Blass Games

- The dual of a game is simply a swapping of roles
- Disjunctions follow by de Morgan
- Given game interpretation of atomics $P \mapsto G_P$ extend to game interpretation G_A for all formulas

Theorem (Blass,1992)

*A is provable in affine logic $\Rightarrow$ **P** has winning strategy in G_A*

(Completeness only for additive fragment)

Blass Games

- The dual of a game is simply a swapping of roles
- Disjunctions follow by de Morgan
- Given game interpretation of atomics $P \mapsto G_P$ extend to game interpretation G_A for all formulas

Theorem (Blass,1992)

*A is provable in affine logic $\Rightarrow$ **P** has winning strategy in G_A*
(Completeness only for additive fragment)

- **Abramsky and Jagadeesan'1992**
  Soundness and completeness for MLL + mix rule
- **Hyland and Ong'1993**
  Soundness and completeness for MLL

Outline

1. Brief Overview
   - Hintikka games (Classical Logic)
   - Lorenzen games (Intuitionistic Logic)
   - Blass games (Linear Logic)
2. Functional Interpretations
   - Strategies as moves
   - Realizability and dialectica
3. Quantifiers and Selection Functions
   - von Neumann games
   - A generalization
   - Interpreting countable and dependent choice

> It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.

Hintikka and Kulas, *The Game of Language*, 1983

> It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.

Hintikka and Kulas, *The Game of Language*, 1983

> Our category of games is a special case of a general construction in the appendix to Barr's book [1]. It is closely related to de Paiva's *dialectica categories* [10,11].

Lafont/Streicher, *Games semantics for LL*, 1991

It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.

Hintikka and Kulas, *The Game of Language*, 1983

Our category of games is a special case of a general construction in the appendix to Barr's book [1]. It is closely related to de Paiva's *dialectica categories* [10,11].

Lafont/Streicher, *Games semantics for LL*, 1991

In developing a category-theoretic approach to the Dialectica interpretation, de Paiva [3] found a connection with linear logic. This connection suggests looking at the Dialectica interpretation, in de Paiva's category-theoretic version, from the point of view of game semantics, and this is the purpose of the present section.

Blass, *A game semantics for LL*, 1992

Functional Moves

What if we could allow for higher-order moves?

Functional Moves

What if we could allow for higher-order moves?

Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \quad \Rightarrow \quad \exists f \forall x Q(x, fx)$$

Functional Moves

What if we could allow for higher-order moves?

Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \quad \Rightarrow \quad \exists f \forall x Q(x, fx)$$

Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into **two-round games**

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

Functional Moves

What if we could allow for higher-order moves?

Can make use of Skolemisation

$$\forall x \exists y Q(x, y) \quad \Rightarrow \quad \exists f \forall x Q(x, fx)$$

Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into **two-round games**

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

P chooses $t = \langle t_0 \ldots t_n \rangle$, then **O** chooses $s = \langle s_0 \ldots s_n \rangle$

P wins iff $Q(s_0, t_0(s_0), \ldots, s_n, t_n(\vec{s}))$

Finite Types and System T

Types generated by

$$X, Y \ :\equiv \ \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \uplus Y \mid Y^X$$

Finite Types and System T

Types generated by

$$X, Y \;:\equiv\; \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \uplus Y \mid Y^X$$

Gödel primitive recursor

$$\mathsf{R}(x, f, n) \stackrel{X}{=} \begin{cases} x & \text{if } n = 0 \\ f(n-1, \mathsf{R}(x, f, n-1)) & \text{if } n > 0 \end{cases}$$

where X is an **any finite type**

Finite Types and System T

Types generated by

$$X, Y \;:\equiv\; \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \uplus Y \mid Y^X$$

Gödel primitive recursor

$$\mathsf{R}(x, f, n) \stackrel{X}{=} \begin{cases} x & \text{if } n = 0 \\ f(n-1, \mathsf{R}(x, f, n-1)) & \text{if } n > 0 \end{cases}$$

where X is an **any finite type**

Gödel's system T: Primitive recursive functionals

Finite Types and System T

Types generated by

$$X, Y \;:\equiv\; \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \uplus Y \mid Y^X$$

Gödel primitive recursor

$$\mathsf{R}(x, f, n) \stackrel{X}{=} \begin{cases} x & \text{if } n = 0 \\ f(n-1, \mathsf{R}(x, f, n-1)) & \text{if } n > 0 \end{cases}$$

where X is an **any finite type**

Gödel's system T: Primitive recursive functionals

Remark: Ackermann function definable using $X = \mathbb{N}^{\mathbb{N}}$

Higher-order Games

Each formula A is assigned a **decidable** adjudication relation

$$|A|_y^x \subseteq X \times Y$$

where X, Y are finite types

Higher-order Games

Each formula A is assigned a **decidable** adjudication relation

$$|A|^x_y \subseteq X \times Y$$

where X, Y are finite types

Intuition:

- **P** plays first choosing t^X
- **O** then chooses s^Y
- **P** wins iff $|A|^t_s$ holds (provable in **T**)

Higher-order Games

Each formula A is assigned a **decidable** adjudication relation

$$|A|^x_y \subseteq X \times Y$$

where X, Y are finite types

Intuition:

- **P** plays first choosing t^X
- **O** then chooses s^Y
- **P** wins iff $|A|^t_s$ holds (provable in **T**)

Theorem (Gödel, 1958)

$$\mathsf{HA} \vdash A \quad \stackrel{\exists t \in \mathbf{T}}{\Longrightarrow} \quad \mathbf{T} \vdash \forall y |A|^t_y$$

Higher-order Games

Turning every formula into $\exists\forall$-form.

Higher-order Games

Turning every formula into ∃∀-form. Assume

$$A \ \mapsto \ \exists x^X \forall y |A|_y^x \qquad\qquad B \ \mapsto \ \exists v^V \forall w |B|_w^v$$

Higher-order Games

Turning every formula into ∃∀-form. Assume

$$A \;\mapsto\; \exists x^X \forall y |A|_y^x \qquad\qquad B \;\mapsto\; \exists v^V \forall w |B|_w^v$$

For instance:

$$A \wedge B \quad \mapsto \quad \exists \langle x, v\rangle \forall \langle y, w\rangle (|A|_y^x \wedge |B|_w^v)$$

Higher-order Games

Turning every formula into $\exists\forall$-form. Assume

$$A \;\mapsto\; \exists x^X \forall y |A|^x_y \qquad\qquad B \;\mapsto\; \exists v^V \forall w |B|^v_w$$

For instance:

$$A \wedge B \quad\mapsto\quad \exists \langle x, v\rangle \forall \langle y, w\rangle (|A|^x_y \wedge |B|^v_w)$$

$$A \vee B \quad\mapsto\quad \exists z^{X \uplus V} \forall \langle y, w\rangle \left\{ \begin{array}{ll} |A|^x_y & \text{if } z = \mathsf{inj}_l(x) \\ |B|^v_w & \text{if } z = \mathsf{inj}_r(v) \end{array} \right\}$$

Higher-order Games

Turning every formula into ∃∀-form. Assume

$$A \mapsto \exists x^X \forall y |A|^x_y \qquad\qquad B \mapsto \exists v^V \forall w |B|^v_w$$

For instance:

$$A \wedge B \quad \mapsto \quad \exists \langle x, v\rangle \forall \langle y, w\rangle (|A|^x_y \wedge |B|^v_w)$$

$$A \vee B \quad \mapsto \quad \exists z^{X \uplus V} \forall \langle y, w\rangle \left\{ \begin{array}{ll} |A|^x_y & \text{if } z = \text{inj}_l(x) \\ |B|^v_w & \text{if } z = \text{inj}_r(v) \end{array} \right\}$$

$$A \to B \quad \mapsto \quad \exists x \forall y |A|^x_y \to \exists v \forall w |B|^v_w$$

Higher-order Games

Turning every formula into $\exists\forall$-form. Assume

$$A \mapsto \exists x^X \forall y |A|_y^x \qquad B \mapsto \exists v^V \forall w |B|_w^v$$

For instance:

$$\begin{aligned} A \wedge B &\mapsto \exists \langle x, v\rangle \forall \langle y, w\rangle (|A|_y^x \wedge |B|_w^v) \\ A \vee B &\mapsto \exists z^{X \uplus V} \forall \langle y, w\rangle \left\{ \begin{array}{ll} |A|_y^x & \text{if } z = \mathsf{inj}_l(x) \\ |B|_w^v & \text{if } z = \mathsf{inj}_r(v) \end{array} \right\} \\ A \rightarrow B &\mapsto \exists x \forall y |A|_y^x \rightarrow \exists v \forall w |B|_w^v \\ &\mapsto \forall x \exists v \forall w \exists y (|A|_y^x \rightarrow |B|_w^v) \end{aligned}$$

Higher-order Games

Turning every formula into $\exists\forall$-form. Assume

$$A \;\mapsto\; \exists x^X \forall y |A|^x_y \qquad\qquad B \;\mapsto\; \exists v^V \forall w |B|^v_w$$

For instance:

$$A \wedge B \quad\mapsto\quad \exists \langle x, v\rangle \forall \langle y, w\rangle (|A|^x_y \wedge |B|^v_w)$$

$$A \vee B \quad\mapsto\quad \exists z^{X \uplus V} \forall \langle y, w\rangle \left\{ \begin{array}{ll} |A|^x_y & \text{if } z = \mathsf{inj}_l(x) \\ |B|^v_w & \text{if } z = \mathsf{inj}_r(v) \end{array} \right\}$$

$$\begin{array}{rcl} A \to B & \mapsto & \exists x \forall y |A|^x_y \to \exists v \forall w |B|^v_w \\ & \mapsto & \forall x \exists v \forall w \exists y (|A|^x_y \to |B|^v_w) \\ & \mapsto & \exists \langle f, g\rangle \forall \langle x, w\rangle (|A|^x_{gxw} \to |B|^{fx}_w) \end{array}$$

Higher-order Games

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$|A \wedge B|^{\langle x,v \rangle}_{\langle y,w \rangle} \quad \equiv \quad |A|^x_y \wedge |B|^v_w$$

Higher-order Games

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$|A \wedge B|_{\langle y,w\rangle}^{\langle x,v\rangle} \equiv |A|_y^x \wedge |B|_w^v$$

$$|A \vee B|_{\langle y,w\rangle}^{\mathsf{inj}_b x} \equiv \begin{cases} |A|_y^x & \text{if } b = l \\ |B|_w^x & \text{if } b = r \end{cases}$$

Higher-order Games

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$
\begin{aligned}
|A \wedge B|^{\langle x,v\rangle}_{\langle y,w\rangle} &\equiv |A|^x_y \wedge |B|^v_w \\
|A \vee B|^{\mathsf{inj}_b x}_{\langle y,w\rangle} &\equiv \begin{cases} |A|^x_y & \text{if } b = l \\ |B|^x_w & \text{if } b = r \end{cases} \\
|A \to B|^{\langle f,g\rangle}_{\langle x,w\rangle} &\equiv |A|^x_{gxw} \to |B|^{fx}_w
\end{aligned}
$$

Higher-order Games

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$
\begin{aligned}
|A \wedge B|^{\langle x,v\rangle}_{\langle y,w\rangle} &\equiv |A|^x_y \wedge |B|^v_w \\
|A \vee B|^{\mathsf{inj}_b x}_{\langle y,w\rangle} &\equiv \begin{cases} |A|^x_y & \text{if } b = l \\ |B|^x_w & \text{if } b = r \end{cases} \\
|A \to B|^{\langle f,g\rangle}_{\langle x,w\rangle} &\equiv |A|^x_{gxw} \to |B|^{fx}_w \\
|\exists z A|^{\langle a,x\rangle}_y &\equiv |A[a/z]|^x_y
\end{aligned}
$$

Higher-order Games

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$|A \wedge B|^{\langle x,v \rangle}_{\langle y,w \rangle} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{\mathsf{inj}_b x}_{\langle y,w \rangle} \equiv \begin{cases} |A|^x_y & \text{if } b = l \\ |B|^x_w & \text{if } b = r \end{cases}$$

$$|A \to B|^{\langle f,g \rangle}_{\langle x,w \rangle} \equiv |A|^x_{gxw} \to |B|^{fx}_w$$

$$|\exists z A|^{\langle a,x \rangle}_y \equiv |A[a/z]|^x_y$$

$$|\forall z A|^f_{\langle a,y \rangle} \equiv |A[a/z]|^{fa}_y$$

Functional interpretations

Higher-order game above is Gödel's **dialectica interpretation**

$$\boxed{|A|^x_y \;\equiv\; A_D(x;y)}$$

Functional interpretations

Higher-order game above is Gödel's **dialectica interpretation**

$$\boxed{|A|^x_y \equiv A_D(x;y)}$$

Variant where interpretation of implication is changed

$$|A \to B|^f_{\langle x,w\rangle} \equiv \forall y |A|^x_y \to |B|^{fx}_w$$

gives Kreisel's **modified realizability**

$$\boxed{\forall y |A|^x_y \equiv x \,\mathbf{mr}\, A}$$

Functional interpretations

Higher-order game above is Gödel's **dialectica interpretation**

$$\boxed{|A|^x_y \;\equiv\; A_D(x;y)}$$

Variant where interpretation of implication is changed

$$|A \to B|^f_{\langle x,w\rangle} \;\equiv\; \forall y |A|^x_y \to |B|^{fx}_w$$

gives Kreisel's **modified realizability**

$$\boxed{\forall y |A|^x_y \;\equiv\; x \ \mathbf{mr} \ A}$$

In either case we have:

If A *is provable in* HA *then* **P** *has winning move in game* $|A|$

Functional interpretations – Completeness

No completeness! Extra principles validated:

AC $\forall x \exists y A(x, y) \to \exists f \forall x A(x, fx)$

MP $\neg\neg\exists x P(x) \to \exists x P(x)$

IP $(A_\forall \to \exists x B(x)) \to \exists x(A_\forall \to B(x))$

Functional interpretations – Completeness

No completeness! Extra principles validated:

AC	$\forall x \exists y A(x,y) \to \exists f \forall x A(x, fx)$
MP	$\neg\neg\exists x P(x) \to \exists x P(x)$
IP	$(A_\forall \to \exists x B(x)) \to \exists x (A_\forall \to B(x))$

Theorem

$\mathrm{HA}^\omega + \mathrm{AC} + \mathrm{MP} + \mathrm{IP} \vdash A$ *iff* **P** *has winning move in* $|A|$

Functional interpretations – Completeness

No completeness! Extra principles validated:

AC $\forall x \exists y A(x, y) \to \exists f \forall x A(x, fx)$

MP $\neg\neg\exists x P(x) \to \exists x P(x)$

IP $(A_\forall \to \exists x B(x)) \to \exists x (A_\forall \to B(x))$

Theorem

HA^ω + AC + MP + IP $\vdash A$ *iff* **P** *has winning move in* $|A|$

Beneficial as it gives:

- Prove closure properties
- Way to eliminate such principles from a proof
- Extract computational information from classical proofs

Functional interpretations – Linear logic

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$
\begin{array}{lcl}
|A \,\&\, B|^{\langle x,v\rangle}_{\mathsf{inj}_b y} & \equiv & \begin{cases} |A|^x_y & \text{if } b = 0 \\ |B|^v_y & \text{if } b = 1 \end{cases} \\
|A \otimes B|^{\langle x,v\rangle}_{\langle f,g\rangle} & \equiv & |A|^x_{fv} \wedge |B|^v_{gx} \\
|\forall z A|^f_{\langle a,y\rangle} & \equiv & |A[a/z]|^{fa}_y \\
|A^\perp|^x_y & \equiv & \neg |A|^y_x \\
|!A|^x_f & \equiv & |A|^x_{fx}
\end{array}
$$

Functional interpretations – Linear logic

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$
\begin{array}{lcl}
|A \,\&\, B|^{\langle x,v\rangle}_{\mathsf{inj}_b y} & \equiv & \begin{cases} |A|^x_y & \text{if } b = 0 \\ |B|^v_y & \text{if } b = 1 \end{cases} \\
|A \otimes B|^{\langle x,v\rangle}_{\langle f,g\rangle} & \equiv & |A|^x_{fv} \wedge |B|^v_{gx} \\
|\forall z A|^f_{\langle a,y\rangle} & \equiv & |A[a/z]|^{fa}_y \\
|A^\perp|^x_y & \equiv & \neg |A|^y_x \\
|!A|^x_f & \equiv & |A|^x_{fx} \qquad \text{(Gödel dialectica)}
\end{array}
$$

Functional interpretations – Linear logic

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$
\begin{array}{lcll}
|A \,\&\, B|^{\langle x,v\rangle}_{\mathsf{inj}_b y} & \equiv & \begin{cases} |A|^x_y & \text{if } b = 0 \\ |B|^v_y & \text{if } b = 1 \end{cases} & \\
|A \otimes B|^{\langle x,v\rangle}_{\langle f,g\rangle} & \equiv & |A|^x_{fv} \wedge |B|^v_{gx} & \\
|\forall z A|^f_{\langle a,y\rangle} & \equiv & |A[a/z]|^{fa}_y & \\
|A^\perp|^x_y & \equiv & \neg |A|^y_x & \\
|!A|^x_f & \equiv & |A|^x_{fx} & \text{(Gödel dialectica)} \\
 & \text{or} & \forall y \in fx\, |A|^x_y & \text{(Diller-Nahm variant)}
\end{array}
$$

Functional interpretations – Linear logic

Assume $|A| \subseteq X \times Y$ and $|B| \subseteq V \times W$ defined. Then:

$$|A \,\&\, B|^{\langle x,v\rangle}_{\mathsf{inj}_b y} \equiv \begin{cases} |A|^x_y & \text{if } b = 0 \\ |B|^v_y & \text{if } b = 1 \end{cases}$$

$$|A \otimes B|^{\langle x,v\rangle}_{\langle f,g\rangle} \equiv |A|^x_{fv} \wedge |B|^v_{gx}$$

$$|\forall z A|^f_{\langle a,y\rangle} \equiv |A[a/z]|^{fa}_y$$

$$|A^\perp|^x_y \equiv \neg |A|^y_x$$

$$|!A|^x_f \equiv |A|^x_{fx} \quad \text{(Gödel dialectica)}$$

$$\text{or} \quad \forall y \in fx\, |A|^x_y \quad \text{(Diller-Nahm variant)}$$

$$\text{or} \quad \forall y |A|^x_y \quad \text{(Kreisel m. realizability)}$$

Outline

1. Brief Overview
   - Hintikka games (Classical Logic)
   - Lorenzen games (Intuitionistic Logic)
   - Blass games (Linear Logic)

2. Functional Interpretations
   - Strategies as moves
   - Realizability and dialectica

3. Quantifiers and Selection Functions
   - von Neumann games
   - A generalization
   - Interpreting countable and dependent choice

von Neumann (Sequential) Games

- n players $\{1, 2, \ldots, n\}$ playing **sequentially**

von Neumann (Sequential) Games

- n players $\{1, 2, \ldots, n\}$ playing **sequentially**
- each player i chooses his move from a set X_i

von Neumann (Sequential) Games

- n players $\{1, 2, \ldots, n\}$ playing **sequentially**
- each player i chooses his move from a set X_i
- a **play** of the game is a sequence $\vec{x} \in X_1 \times \ldots \times X_n$

von Neumann (Sequential) Games

- n players $\{1, 2, \ldots, n\}$ playing **sequentially**
- each player i chooses his move from a set X_i
- a **play** of the game is a sequence $\vec{x} \in X_1 \times \ldots \times X_n$
- payoff function $q\colon \underbrace{X_1 \times \ldots \times X_n}_{\text{play}} \to \underbrace{\mathbb{R}^n}_{\text{payoff}}$

von Neumann (Sequential) Games

- n players $\{1, 2, \ldots, n\}$ playing **sequentially**
- each player i chooses his move from a set X_i
- a **play** of the game is a sequence $\vec{x} \in X_1 \times \ldots \times X_n$
- payoff function $q \colon \underbrace{X_1 \times \ldots \times X_n}_{\text{play}} \to \underbrace{\mathbb{R}^n}_{\text{payoff}}$
- each player trying to maximise his own payoff

Strategies and Nash Equlibrium

- **strategy** for player i is a mapping

$$\mathsf{next}_i \colon X_1 \times \ldots \times X_{i-1} \to X_i$$

Strategies and Nash Equlibrium

- **strategy** for player i is a mapping

$$\text{next}_i \colon X_1 \times \ldots \times X_{i-1} \to X_i$$

- **strategy profile** is a tuple $(\text{next}_i)_{1 \leq i \leq n}$

Strategies and Nash Equlibrium

- **strategy** for player i is a mapping

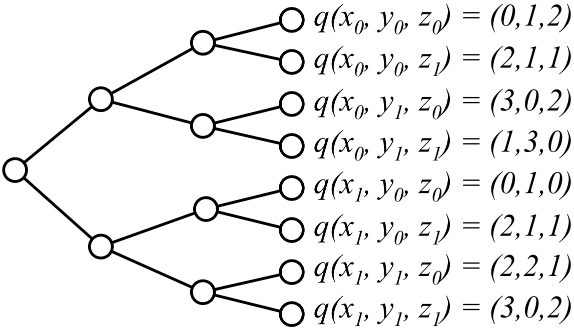
$$\text{next}_i : X_1 \times \ldots \times X_{i-1} \to X_i$$

- **strategy profile** is a tuple $(\text{next}_i)_{1 \leq i \leq n}$
- A strategy profile is in (Nash) **equilibrium** if no single player has an incentive to unilaterally change his strategy

Backward Induction

Three players, payoff function $q\colon X \times Y \times Z \to \mathbb{R}^3$

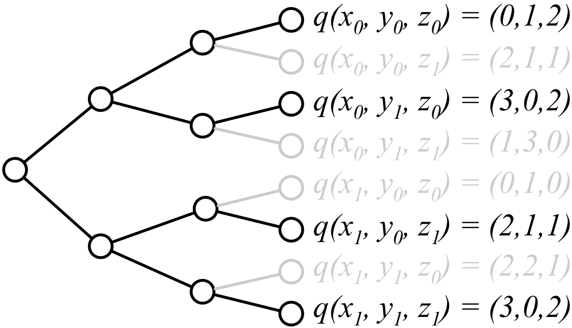
Each player is trying to maximise their own payoff



Backward Induction

Three players, payoff function $q \colon X \times Y \times Z \to \mathbb{R}^3$

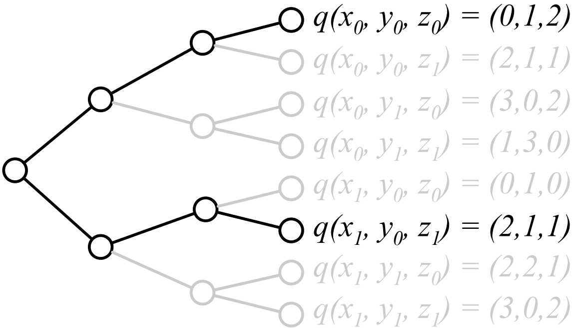
Each player is trying to maximise their own payoff



Backward Induction

Three players, payoff function $q\colon X \times Y \times Z \to \mathbb{R}^3$

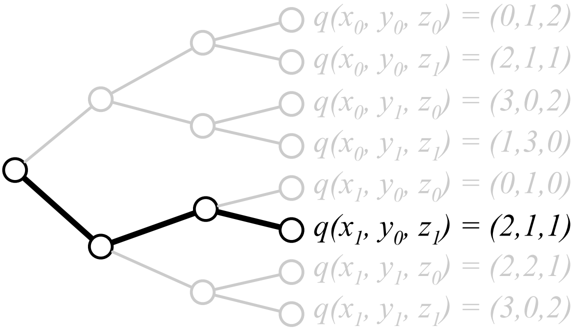
Each player is trying to maximise their own payoff



Backward Induction

Three players, payoff function $q \colon X \times Y \times Z \to \mathbb{R}^3$

Each player is trying to maximise their own payoff



Generalization

We will move from

Player i wants to maximise i-coordinate of payoff

to

Goal at round i is giving by a higher-order function

Quantifiers

For instance:

X = savings accounts

$\mathbb{R}$ = interest paid

Maximise return

$$\max \in (X \to \mathbb{R}) \to \mathbb{R}$$

Quantifiers

For instance:

X = savings accounts

$\mathbb{R}$ = interest paid

Maximise return

$$\max \in (X \to \mathbb{R}) \to \mathbb{R}$$

More generally:

X = set of possible moves

R = set of outcomes

"**Quantifier**"

$$\phi \in \underbrace{(X \to R) \to 2^R}_{K_R X}$$

Quantifiers

For instance:

X = savings accounts

$\mathbb{R}$ = interest paid

Maximise return

$$\max \in (X \to \mathbb{R}) \to \mathbb{R}$$

More generally:

X = set of possible moves

R = set of outcomes

"**Quantifier**"

$$\phi \in \underbrace{(X \to R) \to 2^R}_{K_R X}$$

Other examples: $\exists, \forall, \sup, \int_0^1$, fix, $\ldots$

Quantifiers and Selection Functions

Functionals $\varepsilon\colon \underbrace{(X \to R) \to X}_{J_R X}$ are called **selection functions**

Quantifiers and Selection Functions

Functionals $\varepsilon\colon \underbrace{(X \to R) \to X}_{J_R X}$ are called **selection functions**

A quantifier $\phi : K_R X$ is said to be **attainable** if for some selection function $\varepsilon : J_R X$ we have

$$p(\varepsilon p) \in \phi p$$

for all $p\colon X \to R$

Quantifiers and Selection Functions

Functionals $\varepsilon \colon \underbrace{(X \to R) \to X}_{J_R X}$ are called **selection functions**

A quantifier $\phi : K_R X$ is said to be **attainable** if for some selection function $\varepsilon : J_R X$ we have

$$p(\varepsilon p) \in \phi p$$

for all $p \colon X \to R$

K and J are **strong monads**, so we have $T \in \{K_R, J_R\}$

$$TX \times TY \to T(X \times Y)$$

a **product operation** on selection functions and quantifiers

Quantifiers – von Neumann

For von Neumann "quantifier" at round i is

$$i\text{-max}\colon (X_i \to \mathbb{R}^n) \to 2^{\mathbb{R}^n}$$

defined as

$$i\text{-max}(p) = \{\vec{v} \in \mathbb{R}^n \ : \ \exists x(px = \vec{v}) \wedge \forall x(p_i x \leq v_i)\}$$

Sequential Games – Finite

A **sequential game with** n **rounds** is described by

- Sets of **available moves** X_i for each round $0 \leq i < n$
- A set of **outcomes** R
- **Quantifiers** $\phi_i \colon K_R X_i$ for each round $0 \leq i < n$
- An **outcome function** $q \colon \Pi_{i=0}^{n-1} X_i \to R$

Sequential Games – Unbounded

A **sequential game with** n **rounds** is described by

- Sets of **available moves** X_i for each round $i \in \mathbb{N}$
- A set of **outcomes** R
- **Quantifiers** $\phi_i \colon K_R X_i$ for each round $i \in \mathbb{N}$
- An **outcome function** $q \colon \Pi_{i \in \mathbb{N}} X_i \to R$
- A **clock function** $T \colon \Sigma_n \Pi_{i<n} X_i \to \mathbb{B}$

Sequential Games – Unbounded

A **sequential game with** n **rounds** is described by

- Sets of **available moves** X_i for each round $i \in \mathbb{N}$
- A set of **outcomes** R
- **Quantifiers** $\phi_i \colon K_R X_i$ for each round $i \in \mathbb{N}$
- An **outcome function** $q \colon \Pi_{i \in \mathbb{N}} X_i \to R$
- A **clock function** $T \colon \Sigma_n \Pi_{i<n} X_i \to \mathbb{B}$

We will assume game tree is well-founded

$$\forall \alpha \exists n T(\langle \alpha_0, \ldots, \alpha_n \rangle)$$

Definition (Strategy)

Family of mappings $\mathsf{next}_k \colon \Pi_{i<k} X_i \to X_k$

Definition (Strategy)

Family of mappings $\mathsf{next}_k \colon \Pi_{i<k} X_i \to X_k$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b^{\vec{a}}_k, b^{\vec{a}}_{k+1}, \ldots$ where

$$b^{\vec{a}}_i = \mathsf{next}_i(\vec{a}, b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{i-1})$$

Definition (Strategy)

Family of mappings $\text{next}_k : \Pi_{i<k} X_i \to X_k$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, b_{k+1}^{\vec{a}}, \ldots$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}})$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if

$$q(\vec{a} * \mathbf{b}^{\vec{a}}) \in \phi_k(\lambda x_k . q(\vec{a} * x_k * \mathbf{b}^{\vec{a} * x_k}))$$

for any partial play $\vec{a}$ such that $\neg T(\vec{a})$

Sequential Games – Main Result

Theorem

Fix an unbounded game $G = (X_i, R, \phi_i, q, T)$

Assume $\phi_i \colon K_R X_i$ *attainable with selection fcts* $\varepsilon_i \colon J_R X_i$

Then an optimal strategy for G *can be calculated by an unbounded iterated product of these selection functions as*

$$\mathsf{next}_i(\vec{x}) = \left(\left(\bigotimes_{\vec{x}}^{T} \varepsilon \right) (q) \right)_0$$

Now, what does this have

to do with proof theory?

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg \exists x \forall y |A_n(x)|_y \to \neg\neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg \exists x \forall y |A_n(x)|_y \to \neg\neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall q, \omega \exists \alpha \forall n \leq \omega \alpha \, |A_n(\alpha n)|_{q\alpha}$$

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \; : \; \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg \exists x \forall y |A_n(x)|_y \to \neg\neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall q, \omega \exists \alpha \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha}$$

Finally

$$\forall \varepsilon, q, \omega \exists \alpha \left(\forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha} \right)$$

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg \exists x \forall y |A_n(x)|_y \to \neg\neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall q, \omega \exists \alpha \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha}$$

Finally

$$\forall \varepsilon, q, \omega \exists \alpha \left(\forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha} \right)$$

quantifier at round n

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg\exists x A_n(x) \to \neg\neg\exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg\exists x \forall y |A_n(x)|_y \to \neg\neg\exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall q, \omega \exists \alpha \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha}$$

Finally

outcome function

$$\forall \varepsilon, q, \omega \exists \alpha \left(\forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha} \right)$$

quantifier at round n

Countable Choice

Let us look at negative translation of countable choice:

$$\mathsf{AC}_0^N \ : \ \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

Assuming interpretation of $A_n(x)$ is $|A_n(x)|_y$ we have

$$\forall n \neg\neg \exists x \forall y |A_n(x)|_y \to \neg\neg \exists \alpha \forall n \forall y |A_n(\alpha n)|_y$$

and then

$$\exists \varepsilon \forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall q, \omega \exists \alpha \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha}$$

Finally

outcome function

$$\forall \varepsilon, q, \omega \exists \alpha \left(\forall n \forall p |A_n(\varepsilon_n p)|_{p(\varepsilon_n p)} \to \forall n \leq \omega\alpha \, |A_n(\alpha n)|_{q\alpha} \right)$$

clock function **quantifier at round n**

Countable Choice

Computational interpretation of AC_0 $\equiv$ Theorem about games

Countable Choice

Computational interpretation of $\mathrm{AC}_0 \;\equiv\;$ Theorem about games

Given $|A_n(x)|_y$ and selection fcts. ε_n define quantifiers

$$\phi_n p \;\equiv\; \{y : \; |A_n(\varepsilon p)|_y\}$$

Countable Choice

Computational interpretation of $\mathsf{AC}_0 \;\equiv\;$ Theorem about games

Given $|A_n(x)|_y$ and selection fcts. ε_n define quantifiers

$$\phi_n p \;\equiv\; \{y : \; |A_n(\varepsilon p)|_y\}$$

Premise of $|\mathsf{AC}_0^N|$ says that ϕ_n are attainable with sel. fcts. ε_n

Countable Choice

Computational interpretation of $\mathrm{AC}_0 \;\equiv\;$ Theorem about games

Given $|A_n(x)|_y$ and selection fcts. ε_n define quantifiers

$$\phi_n p \;\equiv\; \{y \,:\, |A_n(\varepsilon p)|_y\}$$

Premise of $|\mathrm{AC}_0^N|$ says that ϕ_n are attainable with sel. fcts. ε_n

Theorem

Given $\varepsilon_i \colon J_R X_i$ (ϕ_i as above) and $q \colon \Pi_i X_i \to R$ and $\omega \colon \Pi_i X_i \to \mathbb{N}$, define the game (X_i, R, ϕ, q, T) where

$$T(s) \equiv \omega(s * \mathbf{0}) < |s|.$$

If ϕ_i are attainable with selection functions ε_i then there exists an optimal play α in the game

Few References

A. Blass
A game semantics for linear logic
APAL, 56:183-220, 1992

P. Oliva
Unifying functional interpretations
NDJFL,47(2):263-290, 2006

P. Oliva
Hybrid functional interp. of linear and intuitionistic logic
Journal of Logic and Computation, 2010

M. Escardó and P. Oliva
Sequential games and optimal strategies
Proceedings of the Royal Society A, 467:1519-1545, 2011