Queen Mary University of London
School of Electronic Engineering and Computer Science
London, England, E1 4NS

Backward Induction and Higher-order Computability

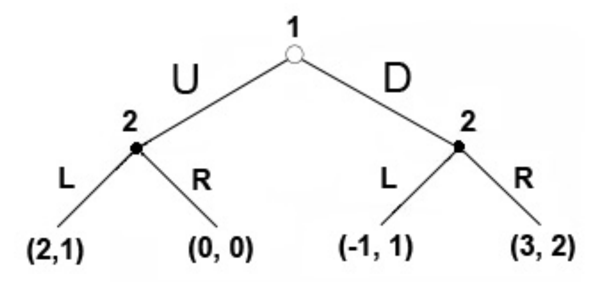
Martín Escardó and Paulo Oliva

Theory Group

paulo.oliva@eecs.qmul.ac.uk
http://www.eecs.qmul.ac.uk/research/theory

1 Backward Induction

The description of a (multi-player) game in *extensive form* is given by a tree of possible plays in the game, with the outcome of each play described in the leaves of the tree.



a. Extensive Form

	(L,L)	(L,R)	(R,L)	(R,R)
U	2,1	2,1	0,0	0,0
D	-1,1	3,2	-1,1	3,2

b. Strategic Form

In sequential games with perfect information a strategy profile consists of a family of mappings

$$\mathsf{next}_i\colon X_1 \times \ldots \times X_{i-1} \to X_i$$

for each round i, calculating the move at point i given what has already been played up to then.

Given a game in extensive form, *backward induction* is a technique used to calculate a strategy profile in sub-game perfect equilibrium.

In a game with n rounds, one can easily calculate the optimal mapping next_n as

$$\mathsf{next}_n(x_1, \ldots, x_{n-1}) = \mathsf{argmax}(\lambda y.q(x_1, \ldots, x_{n-1}, y))$$

where $\mathsf{argmax}\colon (X_n \to \mathbb{R}) \to X_n$ find the point where the function attains its maximum. Once the last optimal strategy is computed, one can then calculate the last-but-one and so on.

2 Selection Functions

We call a *quantifier* any functional of type $(X \to R) \to R$. For instance

$$\begin{array}{lcl} \exists, \forall & : & (X \to \mathbb{B}) \to \mathbb{B} \\ \sup, \inf & : & (X \to \mathbb{R}) \to \mathbb{R} \\ \int & : & (X \to \mathbb{R}) \to \mathbb{R} \\ \mathsf{fix} & : & (X \to X) \to X \end{array}$$

We call a *selection function* any functional of type $(X \to R) \to X$. For instance

$$\begin{array}{lcll} \varepsilon & : & (X \to \mathbb{B}) \to X & \text{(Hilbert's } \varepsilon\text{-term)} \\ \mathrm{argsup}, \mathrm{arginf} & : & (X \to \mathbb{R}) \to X & \\ \mathsf{fix} & : & (X \to X) \to X & \end{array}$$

A quantifier ϕ is *attainable* if for some selection function ε

$$\phi p = p(\varepsilon p)$$

3 Product of Selection Functions

Abbreviate $(X \to R) \to X$ by $J_R X$.

Given two selection functions $\varepsilon\colon J_R X$ and $\delta\colon J_R Y$ we define a new selection function $(\varepsilon \otimes \delta)\colon J_R(X \times Y)$ as

$$(\varepsilon \otimes \delta)(q) = (a, B[a])$$

where $B[x] = \delta(\lambda y.q(x, y))$ and $a = \varepsilon(\lambda x.q(x, B[x]))$.

Given a sequence of selection function $\varepsilon_i\colon J_R X_i$ we define their product as

$$\bigotimes \varepsilon = \varepsilon_1 \otimes (\varepsilon_2 \ldots \otimes (\varepsilon_{n-1} \otimes \varepsilon_n))$$

4 Computing Equilibria

We have shown that the product of selection functions computes sub-game perfect equilibria.

For simplicity consider a single payoff outcome, i.e. assume a given outcome function $q\colon X_1 \ldots X_n \to \mathbb{R}$ where all players are trying to maximise the outcome. They, the optimal play can be calculated as

$$\vec{x} = (\bigotimes \mathrm{argsup})(q)$$

If the outcome is a tuple $\mathbb{R}^n$ where each player is trying to maximise their own payoff we simply have to replace argsup by

$$\mathrm{argsup}_i(p) = \text{any point } x \text{ which maximises } i\text{-th coordinate of } p(x)$$

at each round i.

Finally, a strategy profile in sub-game perfect equilibrium can be calculated as

$$\mathsf{next}_i(x_1, \ldots, x_{i-1}) = ((\mathrm{argsup}_i \otimes \ldots \otimes \mathrm{argsup}_n)(q_{x_1,\ldots,x_{i-1}}))_0$$

5 Unbounded Games

We showed that this also works if one replaces the finite product above by an unbounded product

$$\bigotimes_i \varepsilon = \varepsilon_i \otimes (\bigotimes_{i+1} \varepsilon)$$

Given an infinite sequence of selection functions.

This also involves a generalisation of the games above where quantifiers and selection functions are used to describe the game, rather than the particular case of $\sup$ and argsup.

The unbounded product of selection functions has also been shown to be equivalent to bar recursion (Proof Theory) and to witness a computational version of the Tychnoff theorem (Topology). This shows a new surprising connection between Game Theory, Logic and Mathematics.

References

[1] M. Escardó and P. Oliva. Sequential games and optimal strategies. Proceedings of the Royal Society A, 467:1519-1545, 2011.
[2] M. Escardó and P. Oliva. Selection functions, bar recursion and backward induction. Mathematical Structures in Computer Science, 20(2):127-168, 2010.

Poster developed by Paulo Oliva.
Funded by Royal Society URF fellowship 516002.K501/RH/kk.
Produced using LaTeX.