On Pocrims and Hoops

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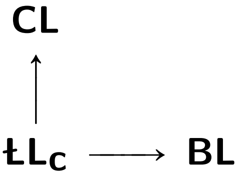
Logics

CL

CL Continuous

Logics



CL	Continuous
ŁL$_C$	Lukasiewicz
BL	Boolean

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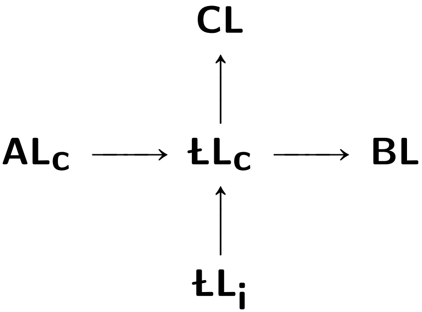

CL	Continuous		
$\mathbf{ŁL_c}$	Lukasiewicz	$\mathbf{ŁL_i}$	Intuitionistic Lukasiewicz
BL	Boolean	$\mathbf{AL_c}$	Affine

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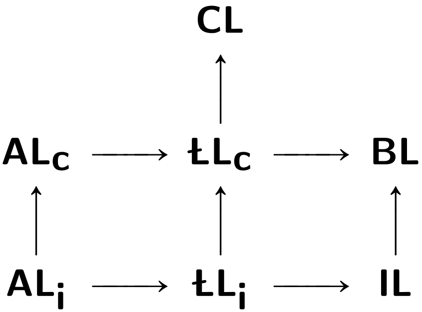

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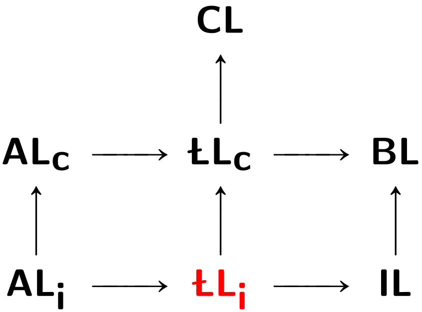

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Algebras

$\mathbf{AL_i}$ Bounded pocrims* $(0, 1, +, \to)$ $x \geqslant y \equiv x \to y = 0$

$\mathbf{AL_c}$ Involutive** pocrims

* partially ordered, commutative, integral monoids

** $x = x^{\perp\perp}$, where $x^{\perp} \equiv x \to 1$

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AL$_c$	Involutive** pocrims	
ŁL$_i$	Bounded hoops (pocrims with divisibility***)	
ŁL$_c$	Involutive hoops $\simeq$ MV algebras	

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Outline

Question: Is A valid in hoops?

Approach 1: Ask prover9 and mace4

Approach 2: Use universal algebra

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Class of hoops is a variety

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Search for proofs and counter-examples

Stark contrast with involutive hoops (**ŁL$_\mathbf{C}$**)
Sound and complete for the unit interval $[0, 1]$

Concrete Questions

Valid in **bounded idempotent pocrims** (i.e. **IL**)

$$(x^{\perp\perp} \to x)^{\perp\perp} = 0$$

$$\begin{aligned}(x \to y)^{\perp} &= x^{\perp\perp} + y^{\perp} \\ (x + y)^{\perp} &= x \to y^{\perp}\end{aligned}$$

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Are these valid in **bounded hoops** (i.e. $\mathbf{ŁL_i}$)?

For instance: $\neg\neg(\neg\neg A \Rightarrow A)$

Short derivation in **intuitionistic logic IL**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha}{\neg\neg A \Rightarrow A} \qquad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\bot}}{\neg A}\,\alpha \qquad [\neg\neg A]_\beta}{\bot}}{A}}{\neg\neg A \Rightarrow A}\,\beta \qquad [\neg(\neg\neg A \Rightarrow A)]_\delta}{\bot}}{\neg\neg(\neg\neg A \Rightarrow A)}\,\delta
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How about **intuitionistic Łukasiewicz logic ŁL$_\mathbf{i}$**?

For instance: $\neg(A \Rightarrow B) \Rightarrow (\neg\neg A \wedge \neg B)$

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Prover9 and Mace4

Pre-linearity

$$((x \to y) \to z) \to ((y \to x) \to z) \to z = 0$$

not valid in hoops (in general)

DEMO!

Prover9 and Mace4

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Mace4: found important counter-examples in a semantic analysis of double negation translations in extensions of $\mathbf{AL_i}$ (see paper)

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\begin{aligned}
x \wedge y &\equiv x + (x \to y) && \text{(\textit{weak conjunction})} \\
x \Rightarrow y &\equiv x \to (x + y) && \text{(\textit{strong implication})} \\
x \vee y &\equiv (x \to y) \to y && \text{(\textit{strong disjunction})}
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End result: 17 "natural" lemmas/theorems

(*natural* = *commutativity, de morgan, associativity, etc*)

Sample of Results

Thm A. The following are valid in all bounded hoops

$$
\begin{aligned}
(x \wedge y)^{\perp} &= x \Rightarrow y^{\perp} \\
(x \Rightarrow y)^{\perp} &= x^{\perp\perp} \wedge y^{\perp} \\
(x \vee y)^{\perp} &= x^{\perp} \wedge y^{\perp}
\end{aligned}
$$

$$
\begin{aligned}
(x + y)^{\perp} &= x \rightarrow y^{\perp} \\
(x \rightarrow y)^{\perp} &= x^{\perp\perp} + y^{\perp}
\end{aligned}
$$

Thm B. Double negation mapping is a hoop endomorphism

$$
\begin{aligned}
(x \rightarrow y)^{\perp\perp} &= x^{\perp\perp} \rightarrow y^{\perp\perp} \\
(x + y)^{\perp\perp} &= x^{\perp\perp} + y^{\perp\perp}
\end{aligned}
$$

However . . .

We **know** the following is valid is all hoops

$$x \Rightarrow (y \Rightarrow z) \quad = \quad (x \wedge y) \Rightarrow z$$

but this has defeated prover9

(*could not find proof after several weeks*)

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Ordinal Sum

Let $\mathbf{S}$ and $\mathbf{F}$ be two hoops

The hoop $\mathbf{S} \frown \mathbf{F}$ (**ordinal sum**) is defined as

- The carrier of $\mathbf{S} \frown \mathbf{F}$ is the union of $\mathbf{S}$ and $\mathbf{F}$ identifying 0
- Extend $+$ such that $s + f = f$ for $s \in \mathbf{S}$ and $f \in \mathbf{F}^*$
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Thm [Blok/Ferreirim'00]
For subdirectly irreducible hoops $\mathbf{H}$ *we have*

- $\mathbf{H} = \mathbf{S} \frown \mathbf{F}$, for some hoops $\mathbf{S}, \mathbf{F}$ with
- $\mathbf{S}$ a subdirectly irreducible involutive hoop
  (hence totally ordered)

Subdirectly Irreducible Hoops

Let $\phi[x_1, \ldots, x_n]$ be an identity in the language of hoops

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Thm. ϕ is valid in the class of all hoops

iff

$\phi[x_1, \ldots, x_n]$ is valid in all hoop $\mathbf{H}$ such that

(1) $\mathbf{H}$ is generated by $x_1, \ldots, x_n$

(2) $\mathbf{H}$ can be expressed as an ordinal sum $\mathbf{S} \frown \mathbf{F}$

(3) $\mathbf{S}$ subdirectly irreducible involutive hoop

(4) $S = \{0\}$ iff $H = \{0\}$

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(4) $S = \{0\}$ iff $H = \{0\}$

Proof. Characterisation of subdirectly irreducible hoops + Birkhoff's theorem on subdirect products

Sample of Results (approach 2)

Thm C. The following is valid in all hoops

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$$\mathrm{idem}(\mathbf{H}) = \{\, x \;|\; x = x + x \,\}$$

$\mathrm{idem}(\mathbf{H})$ is a sub-hoop of $\mathbf{H}$

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Hard part: If x and y are idempotent then so is $x \to y$

Thm D also holds for GBL-algebras by a very different proof (Jipsen/Montagna'05)

References

R. Arthan and P. Oliva
On affine logic and Łukasiewicz logic
arXiv (http://arxiv.org/abs/1404.0570), 2014

R. Arthan and P. Oliva
On pocrims and hoops
arXiv (http://arxiv.org/abs/1404.0816), 2014