Closure of System T under the Bar Recursion Rule

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Outline

1. Spector's bar recursion
2. Schwichtenberg's proof
3. A new (more direct) proof

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Spector's Bar Recursion

(1958) Gödel's Dialectica interpretation of *arithmetic* (system T)

(1962) Spector extends interpretation to *analysis* (T + BR)

(1968) Howard interpretation of bar induction (T + BR)

(1971) Scarpellini shows $\mathcal{C}$ is a model of BR

(1979) Schwichtenberg closure theorem (*low types*)

(1981) Howard's ordinal analysis of BR (*low types*)

(1985) Bezem shows $\mathcal{M}$ is a model of BR

Spector's Bar Recursion (Rule)

Given $s\colon \tau^*$ let $\hat{s}\colon \tau^{\mathbb{N}}$ be the extension of s with 0's

For each pair of types τ, σ, and given G, H and Y

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{otherwise} \end{cases}$$

where

$$\begin{array}{lcl} G & : & \tau^* \to \sigma \\ Y & : & \tau^{\mathbb{N}} \to \mathbb{N} \\ H & : & \tau^* \to (\tau \to \sigma) \to \sigma \end{array}$$

Schwichtenberg's Closure Theorem

Theorem

System T is closed under the bar recursion rule when τ's type level is either 0 or 1

That is, given G, H and Y terms in T, the functional

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{otherwise} \end{cases}$$

is also T definable

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Howard (1968) showed that bar recursion of type ρ can be defined using the bar recursion rule of type $(\mathbb{N} \to \rho) \to \rho$

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it follows that T is not closed under the bar recursion rule for $\tau = (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

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Schwichtenberg's Proof

Published in The Journal of Symbolic Logic (1971)

"On bar recursion of type 0 and 1"

5 pages long (actual proof only two pages long)

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6. Hence, BR can be mimicked by ε_0-ordinal recursion

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4. See BR as a recursion on this tree
5. Define order-preserving embedding of tree into ε_0-ordinals
6. Hence, BR can be mimicked by ε_0-ordinal recursion
7. By Tait, we can find equivalent T definition of $\text{BR}(s)$

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Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant n, BR becomes

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{if } |s| \leq n \end{cases}$$

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Let us refer to this T term as cBR

Proof Idea

Part 1: Show that BR is definable in "general BR"

Part 2: Show that T is closed under "general BR"

(*first part works for any type, second part requires the type restriction*)

General BR

For any *bar* S consider the defining equation

$$\mathsf{gBR}^S(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } S(s) \\ H(s)(\lambda x^\tau . \mathsf{gBR}^S(s * x)) & \text{if } \neg S(s) \end{cases}$$

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Definition

We say that a bar S *secures* $Y : \tau^{\mathbb{N}} \to \mathbb{N}$ if for all s^{τ^*}

$$S(s) \quad \Rightarrow \quad \lambda\beta.Y(s * \beta) \text{ is constant}$$

Part 1: BR definable in general BR

Theorem

Fix $Y : \tau^{\mathbb{N}} \to \mathbb{N}$. *The functional*

$$\lambda G, H, s.\mathsf{BR}^{\tau,\sigma}(G, H, Y)(s)$$

is T-definable in gBR^S, *for any bar* S *securing* Y

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Proof.

Use the bar S to spot when Y becomes constant, then apply the T construction for the case when Y is constant. □

Part 2: Closure of T under gBR rule

Theorem

Fix a T-term $Y : \tau^{\mathbb{N}} \to \mathbb{N}$. For some S securing Y the functional gBR^S is T definable.

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Theorem

Fix a T-term $Y : \tau^{\mathbb{N}} \to \mathbb{N}$. *For some* S *securing* Y *the functional* gBR^S *is* T *definable.*

Proof.

(*Construction*) By induction on Y.

(*Correctness proof*) Use a logical relation to show that the constructed term is indeed equivalent to gBR^S. □

The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ \equiv$ the type of gBR. We will map $\mathbb{N}$ to $\mathbb{N}^\circ$.

Let α be a special variable of type $\mathbb{N} \to \mathbb{N}$ (generic)

$$0^\circ \quad = \quad \lambda G.G$$

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(*H can be fixed at outset, but extra work to remember Y*)

The Construction: Recursor

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can be done by induction hypothesis + primitive recursion

The Correctness Proof

Recall $\mathbb{N}^\circ \equiv$ the type of gBR

Fix H. Define logical relation between T terms

Base case:

$$f^{\mathbb{N}^\circ} \sim_{\mathbb{N}} g^{\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \equiv \exists S \text{ securing } g \text{ such that } f = \mathrm{gBR}^S$$

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and, as usual:

$$\begin{aligned} & f^{\rho_0^\circ \to \rho_1^\circ} \sim_{\rho_0 \to \rho_1} g^{\mathbb{N}^{\mathbb{N}} \to (\rho_0 \to \rho_1)} \\ & \qquad \equiv \forall x^{\rho_0^\circ} \forall y^{\mathbb{N}^{\mathbb{N}} \to \rho_0} (x \sim_{\rho_0} y \to f(x) \sim_{\rho_1} \lambda\alpha.g(\alpha)(y\alpha)) \end{aligned}$$

Main Result

Theorem

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Corollary

Fix $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *in T. Then* $\lambda G, H, s.\mathsf{BR}(G, H, Y)(s)$ *is T definable*

Conclusion

Stronger result:

- Only Y needs to be T definable

More explicit construction:

- Given concrete Y, reasonably easy to find T definition of $\lambda G, H, s.\mathsf{BR}(G, H, Y)(s)$

Easy to calibrate T fragments:

- If Y is T_i then $\lambda G, H, s.\mathsf{BR}(G, H, Y)(s)$ is in T_j, where $j = 1 + \max\{1, \mathrm{level}(\sigma)\} + i$.