Realizability and Dialectica
(*with partial functions*)

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Proofs and Computation
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A TYPE-FREE GÖDEL INTERPRETATION

MICHAEL BEESON

The essential feature by which the present interpretation can be distinguished from Gödel's is this: Gödel's interpretation makes essential use of functionals of higher type. A functional of type $(\sigma, \mathfrak{s})$ must be defined on all type σ functionals and take type $\mathfrak{s}$ functionals for values. Our interpretation, however, uses partially defined functions (better: operations).

We have thus solved the proportion:

X : *Gödel's interpretation* as

Kleene's realizability : *Kreisel's "modified realizability"*.

Plan

- **Realizability** (*formulas as sets*)
- **Dialectica** (*formulas as relations*)
- **Point 1**: Relational view is more general
- **Point 2**: Thoughts on Beeson's question

Realizability

Kleene Realizability

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \underbrace{\{\, n : n \mathrel{\mathrm{r}} A \,\}}_{\text{set of realisers of } A}$$

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Volume 10, Number 4, Dec. 1945

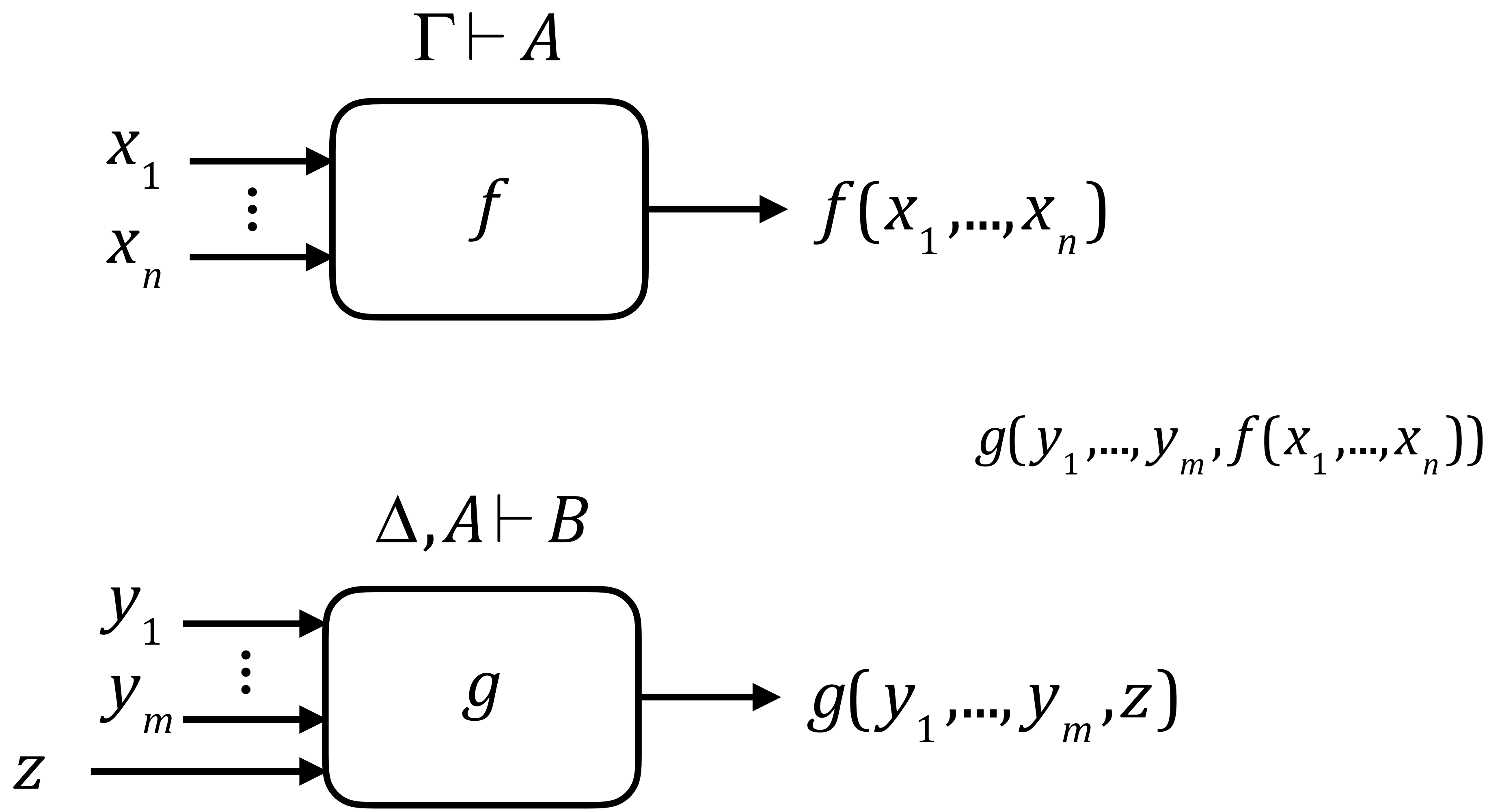
ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

$$
\begin{aligned}
e \,\mathrm{r}\, \mathrm{F} &\equiv e = 0 \wedge \mathrm{F} \\
e \,\mathrm{r}\, A \wedge B &\equiv e_0 \,\mathrm{r}\, A \wedge e_1 \,\mathrm{r}\, B \\
e \,\mathrm{r}\, A \vee B &\equiv (e_0 = 0 \wedge e_1 \,\mathrm{r}\, A) \wedge (e_0 = 1 \wedge e_1 \,\mathrm{r}\, B) \\
e \,\mathrm{r}\, A \to B &\equiv \forall a (a \,\mathrm{r}\, A \to \{e\}(a)\!\downarrow \wedge \{e\}(a) \,\mathrm{r}\, B) \\
e \,\mathrm{r}\, \exists x A(x) &\equiv e_1 \,\mathrm{r}\, A(e_0) \\
e \,\mathrm{r}\, \forall x A(x) &\equiv \forall x (\{e\}(x)\!\downarrow \wedge \{e\}(x) \,\mathrm{r}\, A(x))
\end{aligned}
$$

Theorem (Kleene'45). If HA $\vdash A$ then there exists a numeral n such that HA $\vdash n \; r \; A$

Kleene Realizability



Dialectica

Dialectica Interpretation

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \{\, (x, y) : \underbrace{|A|^{x}_{y}}_{\substack{\text{relation between arguments} \\ \text{and counter-arguments}}} \,\}$$

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$|F \wedge G|^{y,v}_{z,w} \equiv |F|^{y}_{z} \wedge |G|^{v}_{w}$$

$$|F \vee G|^{y,v,t}_{z,w} \equiv (t = 0 \wedge |F|^{y}_{z}) \vee (t = 1 \wedge |G|^{v}_{w})$$

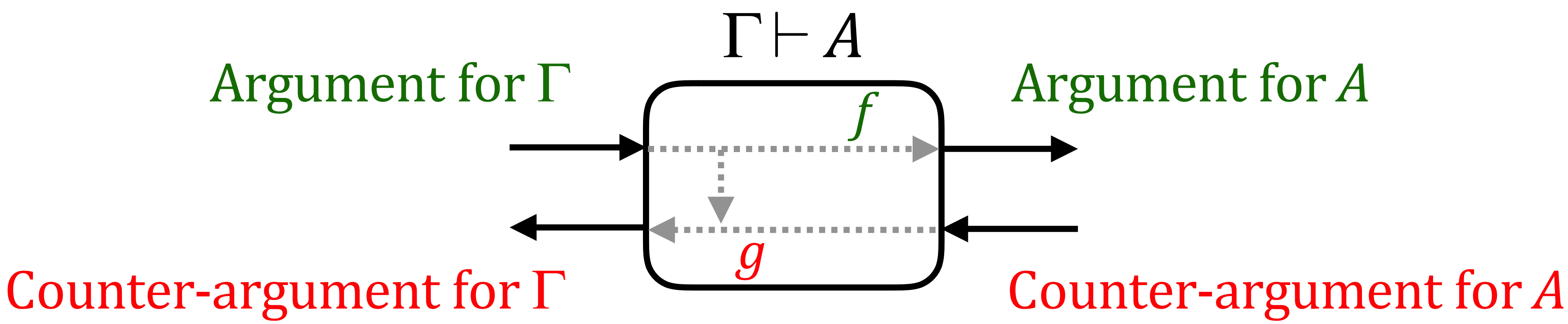
$$|F \to G|^{V,Z}_{y,w} \equiv |F|^{y}_{Z(y,w)} \to |G|^{V(y)}_{w}$$

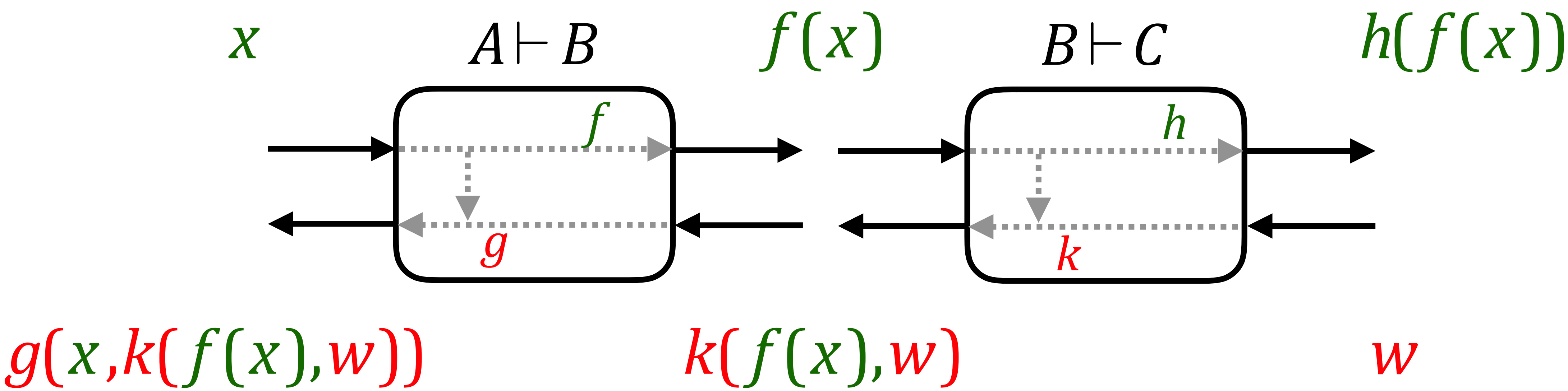
$$|\forall x F(x)|^{Y}_{s,z} \equiv |F(s)|^{Y(s)}_{z}$$

$$|\exists x F(x)|^{s,y}_{z} \equiv |F(s)|^{y}_{z}$$

Theorem (Gödel'58). If HA $\vdash A$ then there exists a term t of system T such that T $\vdash \forall y |A|^{t}_{y}$

Dialectica Interpretation





Point 1:
Kleene realizability in ‘relational’ style

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

Gödel's Dialectica interpretation		
$\lvert F \wedge G \rvert^{y,v}_{z,w}$	$\equiv$	$\lvert F \rvert^{y}_{z} \wedge \lvert G \rvert^{v}_{w}$
$\lvert F \vee G \rvert^{y,v,t}_{z,w}$	$\equiv$	$(t=0 \wedge \lvert F \rvert^{y}_{z}) \vee (t=1 \wedge \lvert G \rvert^{v}_{w})$
$\lvert F \rightarrow G \rvert^{V,Z}_{y,w}$	$\equiv$	$\lvert F \rvert^{y}_{Z(y,w)} \rightarrow \lvert G \rvert^{V(y)}_{w}$
$\lvert \forall s F \rvert^{Y}_{s,z}$	$\equiv$	$\lvert F \rvert^{Y(s)}_{z}$
$\lvert \exists s F \rvert^{s,y}_{z}$	$\equiv$	$\lvert F \rvert^{y}_{z}$

Kreisel's modified realizability		
$\lvert F \wedge G \rvert^{y,v}_{z,w}$	$\equiv$	$\lvert F \rvert^{y}_{z} \wedge \lvert G \rvert^{v}_{w}$
$\lvert F \vee G \rvert^{y,v,t}_{z,w}$	$\equiv$	$(t=0 \wedge \lvert F \rvert^{y}_{z}) \vee (t=1 \wedge \lvert G \rvert^{v}_{w})$
$\lvert F \rightarrow G \rvert^{V}_{y,w}$	$\equiv$	$\forall z \lvert F \rvert^{y}_{z} \rightarrow \lvert G \rvert^{V(y)}_{w}$
$\lvert \forall s F \rvert^{Y}_{s,z}$	$\equiv$	$\lvert F \rvert^{Y(s)}_{z}$
$\lvert \exists s F \rvert^{s,y}_{z}$	$\equiv$	$\lvert F \rvert^{y}_{z}$

$$A^D(x,y) \quad \text{iff} \quad \lvert A \rvert^{x}_{y}$$

$$x \text{ mr } A \quad \text{iff} \quad \forall y \lvert A \rvert^{x}_{y}$$

Kleene numerical realizability

$$
\begin{aligned}
e \,\mathrm{r}\, A \wedge B &\equiv e_0 \,\mathrm{r}\, A \wedge e_1 \,\mathrm{r}\, B \\
e \,\mathrm{r}\, A \vee B &\equiv (e_0 = 0 \wedge e_1 \,\mathrm{r}\, A) \wedge (e_0 = 1 \wedge e_1 \,\mathrm{r}\, B) \\
e \,\mathrm{r}\, A \to B &\equiv \forall a (a \,\mathrm{r}\, A \to \{e\}(a)\downarrow \wedge \{e\}(a) \,\mathrm{r}\, B) \\
e \,\mathrm{r}\, \exists x A(x) &\equiv e_1 \,\mathrm{r}\, A(e_0) \\
e \,\mathrm{r}\, \forall x A(x) &\equiv \forall x (\{e\}(x)\downarrow \wedge \{e\}(x) \,\mathrm{r}\, A(x))
\end{aligned}
$$

Kleene (*dialog*) numerical realizability

$$
\begin{aligned}
|F \wedge G|^e_a &\equiv |F|^{e_0}_{a_0} \wedge |G|^{e_1}_{a_1} \\
|F \vee G|^e_a &\equiv (e_0 = 0 \wedge |F|^{e_1}_a) \vee (e_0 = 1 \wedge |G|^{e_1}_a) \\
|F \to G|^e_a &\equiv \forall b |F|^{a_0}_b \to \{e\}(a_0)\downarrow \wedge |G|^{\{e\}(a_0)}_{a_1} \\
|\exists s F(s)|^e_a &\equiv |F(e_0)|^{e_1}_a \\
|\forall s F(s)|^e_a &\equiv \{e\}(a_0)\downarrow \wedge |F(a_0)|^{\{e\}(a_0)}_{a_1}
\end{aligned}
$$

$$e \,\mathrm{r}\, A \quad \text{iff} \quad \forall a |A|^e_a$$

Point 2:
Dialectica with partial realizers

Kleene realizability

$$|F \to G|^e_a \quad \equiv \quad \forall b\, |F|^{a_0}_b \to \{e\}(a_0)\downarrow \wedge |G|^{\{e\}(a_0)}_{a_1}$$

Attempt 1 (problem interpreting contraction axiom)

$$|F \to G|^e_a \quad \equiv \quad (\{e_1\}(a)\downarrow \to |F|^{a_0}_{\{e_1\}(a)}) \to (\{e_0\}(a_0)\downarrow \wedge |G|^{\{e_0\}(a_0)}_{a_1})$$

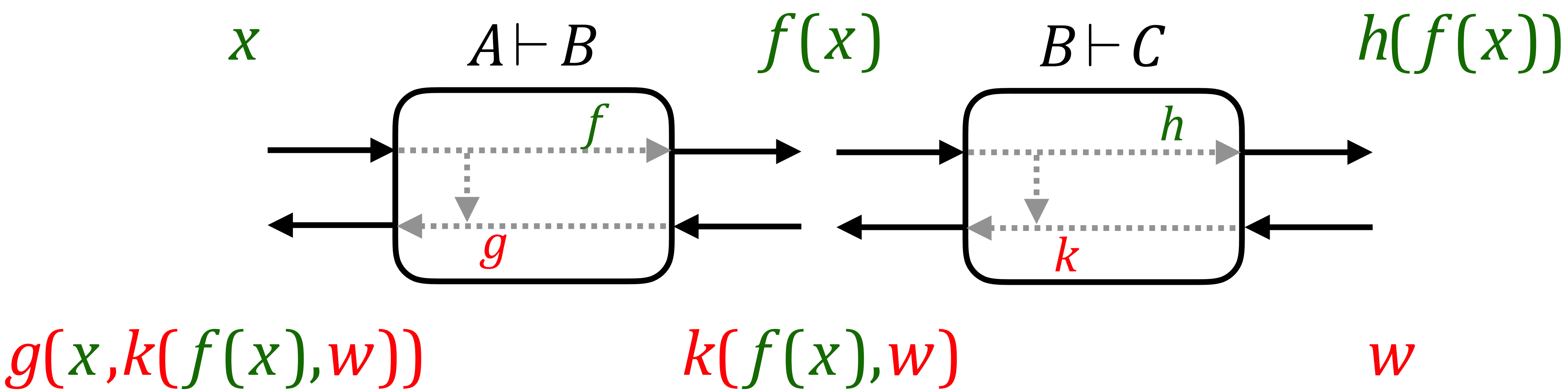
Attempt 2 (still has issue with contraction axiom)

$$|F \to G|^e_a \quad \equiv \quad (\{e_1\}(a)\downarrow \to \forall n \in \{e_1\}(a)\, |F|^{a_0}_n) \to \{e_0\}(a_0)\downarrow \wedge |G|^{\{e_0\}(a_0)}_{a_1})$$

Attempt 3 (problem interpreting composition)

$$|F \to G|^e_a \quad \equiv \quad \{e_1\}(a)\downarrow \wedge (\forall n \in \{e_1\}(a)\, |F|^{a_0}_n \to \{e_0\}(a_0)\downarrow \wedge |G|^{\{e_0\}(a_0)}_{a_1})$$

Issue with Composition



If arguments are allowed to be partial, we can't guarantee counter-arguments are always total

Possible Solution

We need a finite set $W_{\{g\}(x,w)}$ so that we can define:

$$|F \to G|_{x,w}^{f,g} \equiv \forall y \in W_{\{g\}(x,w)} \, |F|_y^x \to \{f\}(x)\downarrow \wedge |G|_w^{\{f\}(x)}$$

But we need to be able to:

- form finite union of W-sets (*contraction*)
- form union of W-sets indexed by W-sets (*composition*)
- W-sets need to be non-empty (*composition*)

Possible Solution

Define W-sets as:

$$W_{e,x} \equiv \{0\} \cup \{ \{e\}(i,x) \mid i \in \mathbb{N} \wedge \{e\}(i,x)\downarrow \}$$

So:

$$\forall y \in W_{e,x}\ A(y) \equiv A(0) \wedge \forall i,u(T(e,(i,x),u) \rightarrow A(U(u))$$

Summary

- Kleene realizability in "relational style"
- When trying to witness counter-argument function
    - Can't witness it precisely because matrix not decidable
    - Can't assume counter-argument functions are total
    - Defined finite sets of possibly undefined elements
    - These sets must be non-empty