On “Approximate” Variants of Realizability and Functional Interpretations

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Facets of Realizability

Cachan, 1 July 2019

“*Far better an approximate answer to the right question than the exact answer to the wrong question.*”

- John Tukey

Plan

1. Realizability and functional interpretations
2. Precise vs approximate
3. Analysis of precise interpretations
4. Analysis of approximate interpretations

1.
Realizability
and
Functional Interpretations

Realizability and Functional Interpretations

formula A $\xrightarrow{I_{\mathcal{F}}}$ *functional specification* $I_{\mathcal{F}}(A)$

proof π_A *of A* $\xrightarrow{I_{\mathcal{P}}}$ *program* $I_{\mathcal{P}}(\pi_A) \in I_{\mathcal{F}}(A)$

formula A $\xrightarrow{I_{\mathcal{F}}}$ *functional specification* $I_{\mathcal{F}}(A)$

$A = \forall n \exists m (m = n^2)$ $\qquad$ $I_{\mathcal{F}}(A) = \langle f;\ n;\ f(n) = n^2 \rangle$

$A = \sqrt{2} \notin \mathbb{Q}$ $\qquad$ $I_{\mathcal{F}}(A) = \langle f;\ p,q;\ |\frac{p}{q} - \sqrt{2}| > f(p,q) \rangle$

$$I_{\mathcal{F}}(A) = \langle f;\, n;\, \underbrace{f(n) = n^2}_{|A|^{f}_{n}} \rangle$$

argument

counter-argument

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

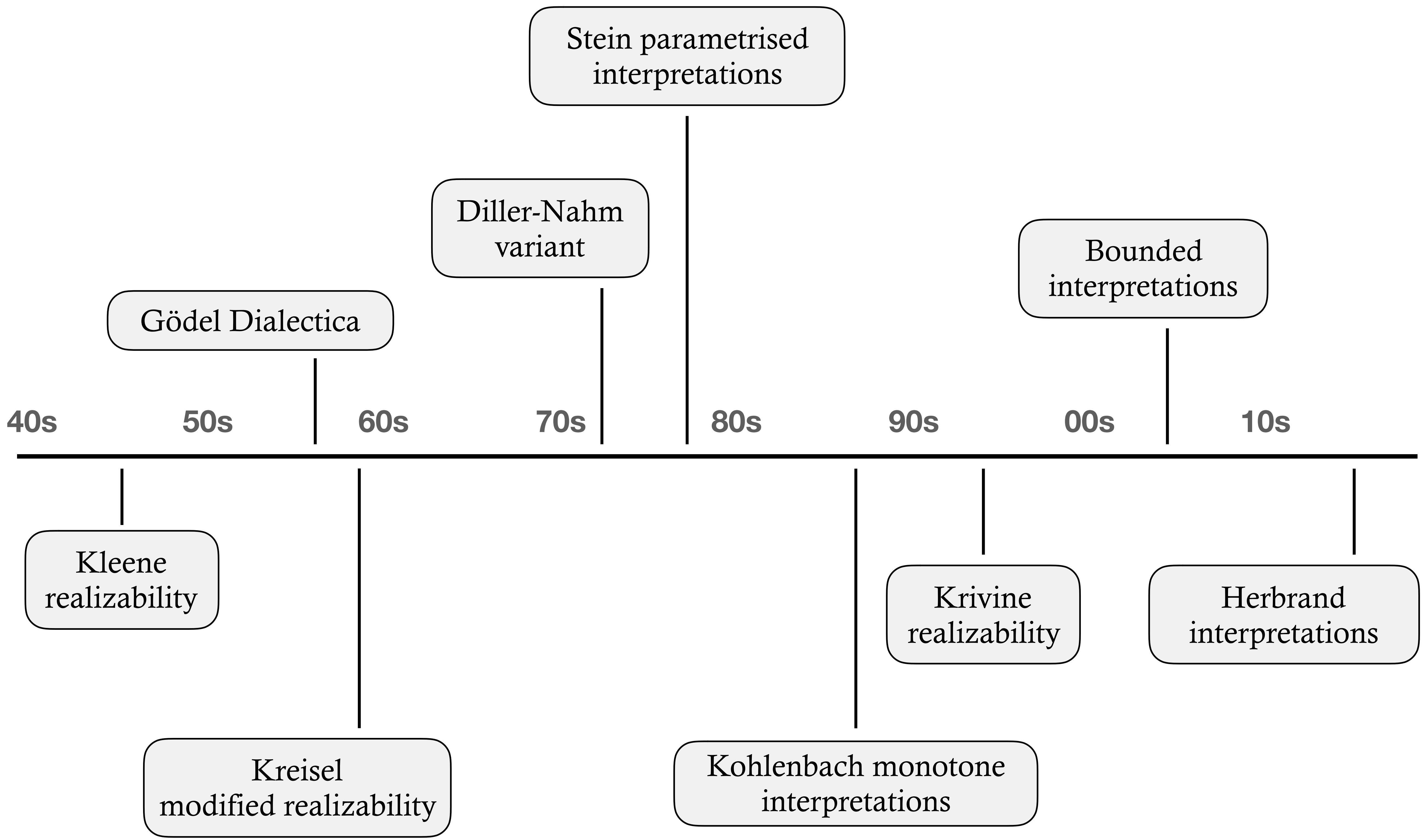
$$|A \vee B|^{x,v,b}_{y,w} \equiv (b=0 \wedge |A|^x_y) \vee (b \neq 0 \wedge |B|^v_w)$$

$$|A \to B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \to |B|^{f(x)}_w$$

$$|\forall z\, A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z\, A(z)|^{x,z}_y \equiv |A(z)|^x_y$$

Theorem (Gödel'58). If HA $\vdash A$ then there exists a term t of system T such that T $\vdash \forall y\, |A|^t_y$


Stein parametrised interpretations
Diller-Nahm variant
Gödel Dialectica
Bounded interpretations
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Kreisel modified realizability
Kohlenbach monotone interpretations
Krivine realizability
Herbrand interpretations

2.
Precise Interpretations
vs
Approximate Interpretations

"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise	**Approximate**
The number is 17	It's an odd number It's less than 50 It's either 11 or 17
The function is $\lambda x . \frac{x}{2}$	The function is bounded by $\lambda x . x^2$ It's either $\lambda x . \frac{x}{2}$ or $\lambda x . x^2$

"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise

$$f(x) = \begin{cases} 0 & \text{if } \exists u.T(x,x,u) \\ 1 & \text{if } \forall u.\neg T(x,x,u) \end{cases}$$

precise

but

non-computable

Approximate

$$g(x) = \{0, 1\}$$

approximate

but

computable

precise inputs

precise outputs

non-computable inputs

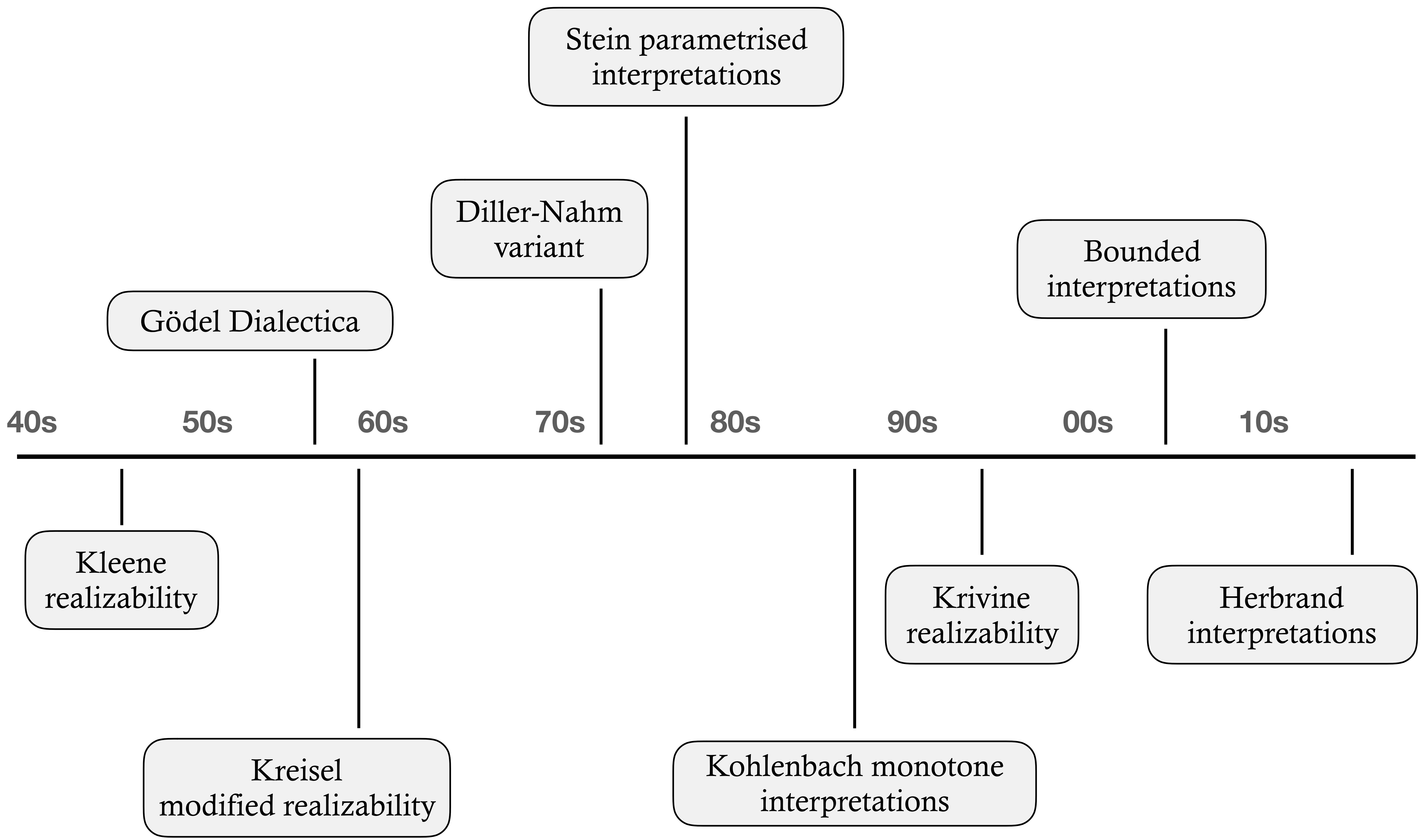
non-computable outputs

approximate inputs

approximate outputs

computable inputs

computable outputs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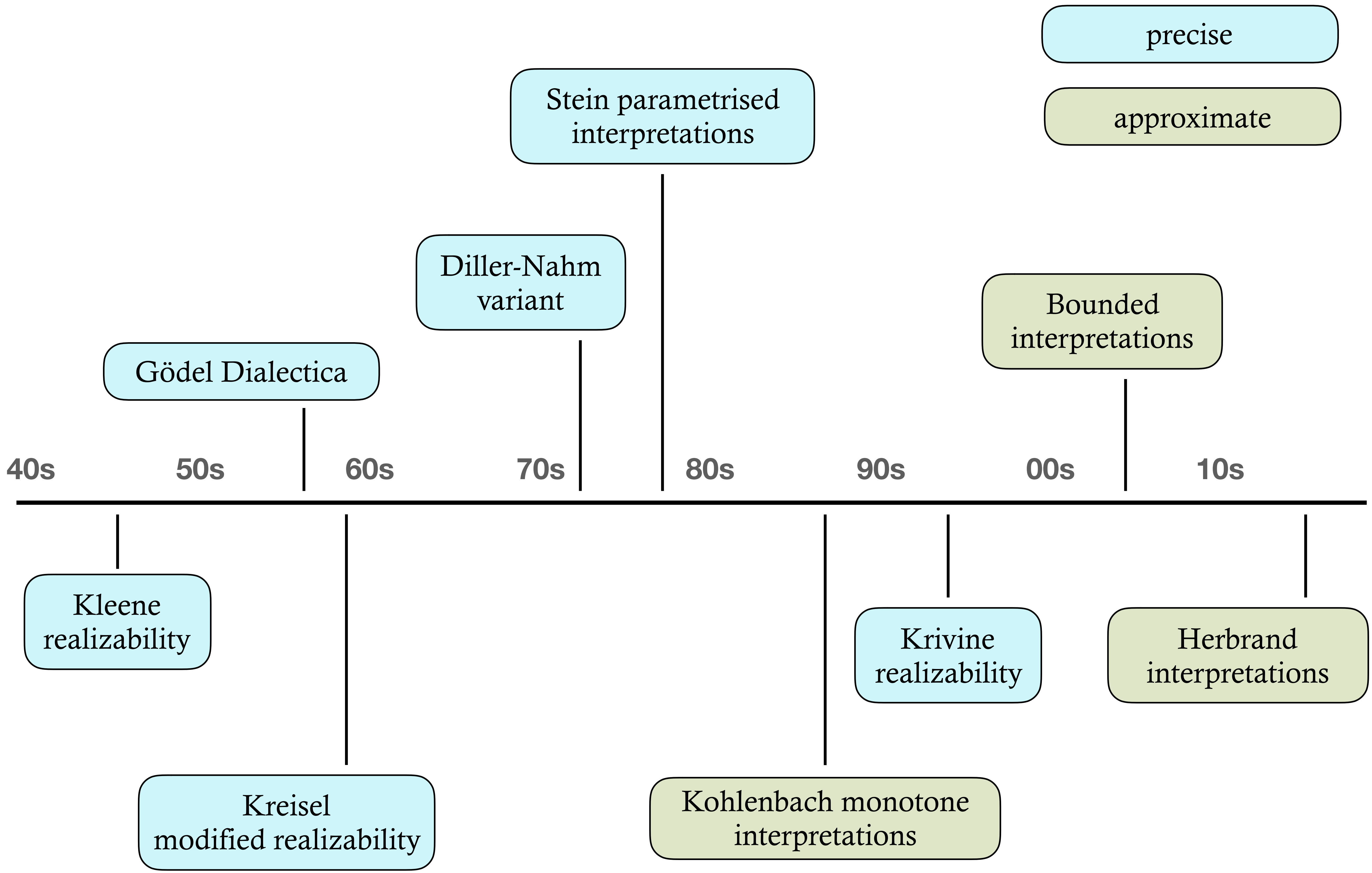
precise
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Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

"Herbrand" interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

What do they have in common?

In which way are they different?

Any others waiting to be discovered?

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

"Herbrand" interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

Result 1: Relational presentation of realizability

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

Result 2: Only differ in the treatment of contraction (!A)

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

Result 3: Multiple exponentials = combined interpretations

M.D. Hernest and P. Oliva, **Hybrid functional interpretations**, CiE, 2008

3.
Unifying Precise Interpretations through Linear Logic

Linear Logic

A refinement of classical and intuitionistic logic

$$A \to B \quad \Rightarrow \quad !A \multimap B$$

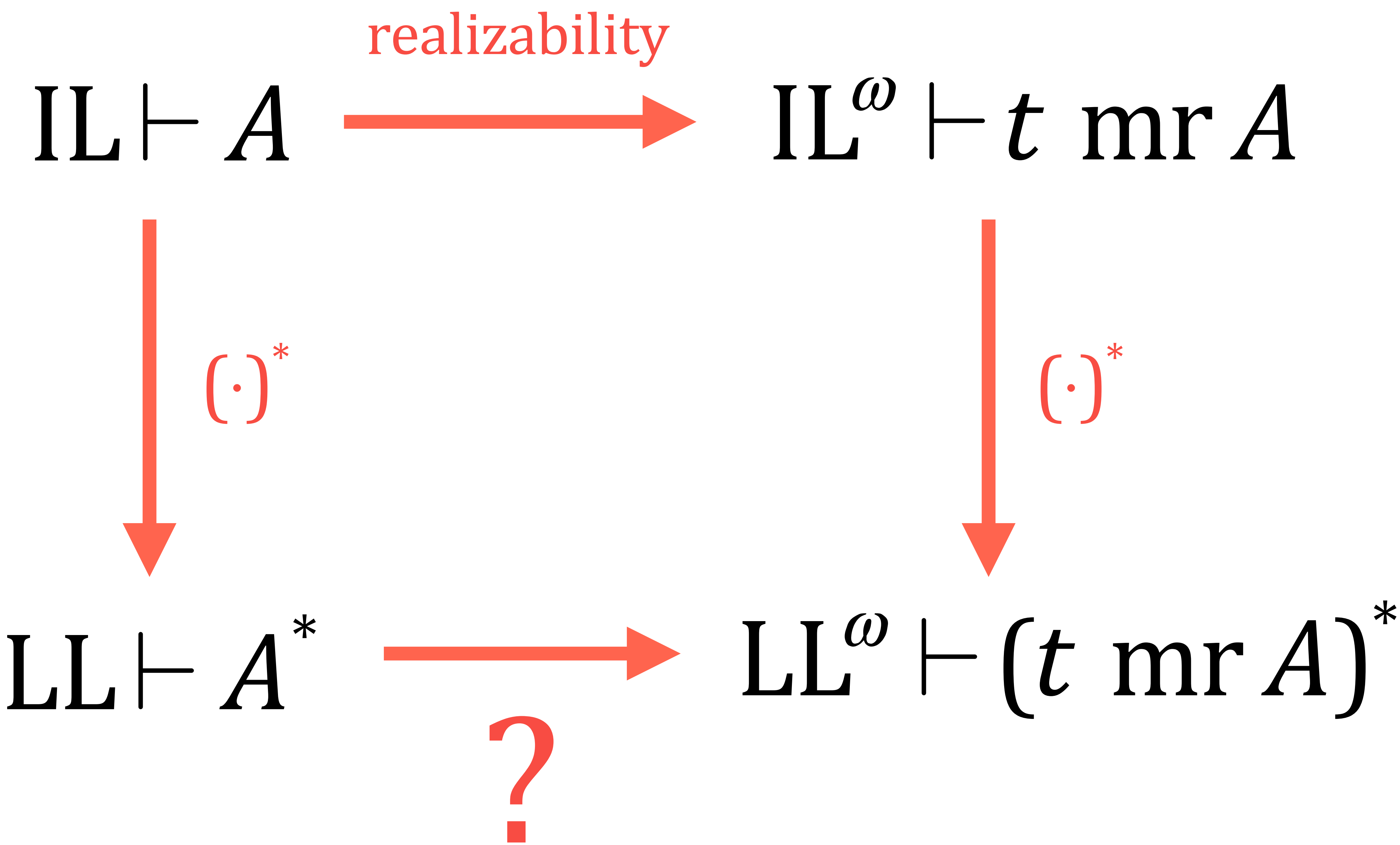
$$A \wedge B \quad \Rightarrow \quad A \,\&\, B$$

$$A \wedge B \quad \Rightarrow \quad A \otimes B$$

call-by-name translation		
$(A \wedge B)^*$	$\equiv$	$A^* \,\&\, B^*$
$(A \vee B)^*$	$\equiv$	$!A^* \oplus !B^*$
$(A \to B)^*$	$\equiv$	$!A^* \multimap B^*$
$(\forall z A)^*$	$\equiv$	$\forall z A^*$
$(\exists z A)^*$	$\equiv$	$\exists z !A^*$

call-by-value translation		
$(A \wedge B)^\circ$	$\equiv$	$A^\circ \otimes B^\circ$
$(A \vee B)^\circ$	$\equiv$	$A^\circ \oplus B^\circ$
$(A \to B)^\circ$	$\equiv$	$!(A^\circ \multimap B^\circ)$
$(\forall z A)^\circ$	$\equiv$	$!\forall z A^\circ$
$(\exists z A)^\circ$	$\equiv$	$\exists z A^\circ$

$$\mathrm{IL} \vdash A \quad \Rightarrow \quad \begin{array}{l} \mathrm{LL} \vdash A^\circ \\ \mathrm{LL} \vdash A^* \end{array}$$


realizability
IL ⊢ A
IL^ω ⊢ t mr A
(·)*
(·)*
LL ⊢ A*
LL^ω ⊢ (t mr A)*
?

Interpretation of Linear Logic

$$
\begin{aligned}
|A \otimes B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \otimes |B|^{v}_{w} \\
|A \oplus B|^{x,v,b}_{y,w} &\equiv (b{=}0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \,\&\, B|^{x,v}_{y,w,b} &\equiv (b{=}0 \,\&\, |A|^{x}_{y}) \oplus (b \neq 0 \,\&\, |B|^{v}_{w}) \\
|A \multimap B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \multimap |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011 modified realizability

$$A \xrightarrow{\text{modified realizability}} x \text{ mr } A$$

$$(\cdot)^* \downarrow \qquad\qquad \downarrow (\cdot)^*$$

$$A^* \longrightarrow (x \text{ mr } A)^* \Leftrightarrow \forall y \, |A^*|^x_y$$

$$|!A|^x \equiv !\forall y \, |A|^x_y$$

interpretations (only) differ in treatment of $!A$

!A	Trans.	Interpretation
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Kreisel modified realizability
$\vert !A\vert^x_a \equiv !\forall y \in a \vert A\vert^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$\vert !A\vert^x_a \equiv !\vert A\vert^x_a$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\vert !A\vert^x \equiv !\forall y \vert A\vert^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$\vert !A\vert^x_a \equiv !\forall y \in a \vert A\vert^x_y \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

4.
Unifying Approximate Interpretations through Linear Logic

Kleene realizability

Gödel Dialectica

Diller-Nahm variant

Bounded interpretations

“Herbrand” interpretations

Kreisel modified realizability

Krivine realizability

Stein parametrised interpretations

Kohlenbach monotone interpretations

How do these fit (if at all) into the “unification”?

Kohlenbach monotone interpretations

formula A $\xrightarrow{I_{\mathcal{F}}}$ *functional specification* $I_{\mathcal{F}}(A)$

proof π_A *of A* $\xrightarrow{I^*_{\mathcal{P}}}$ ***majorant*** $I^*_{\mathcal{P}}(\pi_A) \geq^* I_{\mathcal{F}}(A)$

$Dial_{\mathcal{F}} + Dial^*_{\mathcal{P}}$	=	monotone Dialectica
$Real_{\mathcal{F}} + Real^*_{\mathcal{P}}$	=	monotone modified realizability

Bounded functional interpretation

Fernando Ferreira[a,*], Paulo Oliva[b]

Annals of Pure and Applied Logic 135 (2005) 73–112

$$n \leq^* m \quad \equiv \quad n \leq m$$

$$f \leq^* g \quad \equiv \quad \forall a \forall x \leq^* a (fx \leq^* ga \wedge gx \leq^* ga)$$

$$|A \wedge B|^{x,v}_{y,w} \quad \equiv \quad |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v}_{y,w} \quad \equiv \quad (\forall y' \leq^* y \, |A|^x_{y'}) \vee (\forall w' \leq^* w \, |B|^v_{w'})$$

$$|A \to B|^{f,g}_{x,w} \quad \equiv \quad \forall y \leq^* g(x,w) |A|^x_y \to |B|^{f(x)}_w$$

$$|\forall z A(z)|^f_{y,c} \quad \equiv \quad \forall z \leq^* c \, |A(z)|^{f(c)}_z$$

$$|\exists z A(z)|^{x,c}_y \quad \equiv \quad \exists z \leq^* c \forall y' \leq y \, |A(z)|^x_{y'}$$

A functional interpretation for nonstandard arithmetic

Benno van den Berg[a,*,1], Eyvind Briseid[b,2], Pavol Safarik[c,3]

Annals of Pure and Applied Logic 163 (2012) 1962–1994

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g[x,w]\, |A|^{x}_{y} \to |B|^{f[x]}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \equiv |A(z)|^{f[z]}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \equiv \exists z' \in z \forall y' \in y\, |A(z')|^{x}_{y'}$$

$$f[x] := \bigcup_{f' \in f} f'(x)$$

Bounded functional interpretation

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv (\forall y' \leq^{*} y\, |A|^{x}_{y'}) \vee (\forall w' \leq^{*} w\, |B|^{v}_{w'})$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \leq^{*} g(x,w)\, |A|^{x}_{y} \to |B|^{f(x)}_{w}$$

$$|\forall z\, A(z)|^{f}_{y,c} \equiv \forall z \leq^{*} c\, |A(z)|^{f(c)}_{z}$$

$$|\exists z\, A(z)|^{x,c}_{y} \equiv \exists z \leq^{*} c \forall y' \leq y\, |A(z)|^{x}_{y'}$$

Herbrand functional interpretation

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g[x,w]\, |A|^{x}_{y} \to |B|^{f[x]}_{w}$$

$$|\forall z^{\mathrm{st}} A(z)|^{f}_{y,z} \equiv |A(z)|^{f[z]}_{y}$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_{y} \equiv \exists z' \in z \forall y' \in y\, |A(z')|^{x}_{y'}$$

Result: Herbrand interpretations have alternative presentations that use usual functional application

Herbrand functional interpretation (alternative presentation)

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^x_y \vee |B|^v_w$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g(x,w)\, |A|^x_y \to |B|^{f(x)}_w$$

$$|\forall z^{\mathrm{st}} A(z)|^f_{y,z} \equiv |A(z)|^{f(z)}_y$$

$$|\exists z^{\mathrm{st}} A(z)|^{x,z}_y \equiv \exists z' \in z \forall y' \in y\, |A(z')|^x_{y'}$$

Consequence 1: Arguments are not always sets

Consequence 2: Full monotonicity property no longer holds

$$
\begin{array}{lcl}
(s^{\rho^*})^{(\mathrm{st}^\sigma(z))\uparrow} & :\equiv & s \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A}, \boldsymbol{b}^{\boldsymbol{\tau}^d_B})^{(A\wedge B)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}, (\boldsymbol{b}^{\boldsymbol{\tau}^d_B})^{B\uparrow} \\
(\boldsymbol{\phi}^{\boldsymbol{\tau}^d_A \to \boldsymbol{\tau}^d_B})^{(A\to B)\uparrow} & :\equiv & \{\lambda \boldsymbol{a}^{\boldsymbol{\tau}^u_A}.(\boldsymbol{\phi}(\boldsymbol{a}^{A\downarrow}))^{B\uparrow}\} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A}, c^{\sigma^*})^{(\exists z^{\mathrm{st}^\sigma} A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}, c \\
(\boldsymbol{\phi}^{\sigma^* \to \boldsymbol{\tau}^d_A})^{(\forall z^{\mathrm{st}^\sigma} A)\uparrow} & :\equiv & \{\lambda c^{\sigma^*}.(\boldsymbol{\phi}(c))^{A\uparrow}\} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\exists z^\sigma A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\forall z^\sigma A)\uparrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\uparrow}
\end{array}
$$

For any $\boldsymbol{a} \colon \boldsymbol{\tau}^d_A$ *we have* $(\boldsymbol{a}^{A\uparrow})^{A\downarrow} =_{\boldsymbol{\tau}^d_A} \boldsymbol{a}$

$$
\begin{array}{ll}
(i) & \boldsymbol{a} \ \mathrm{hr}' \ A \ \Rightarrow \ \boldsymbol{a}^{A\uparrow} \ \mathrm{hr} \ A \\
(ii) & \boldsymbol{a} \ \mathrm{hr} \ A \ \Rightarrow \ \boldsymbol{a}^{A\downarrow} \ \mathrm{hr}' \ A
\end{array}
$$

$$
\begin{array}{lcl}
(s^{\rho^*})^{(\mathrm{st}^\sigma(z))\downarrow} & :\equiv & s \\
(\boldsymbol{a}^{\boldsymbol{\tau}^u_A}, \boldsymbol{b}^{\boldsymbol{\tau}^u_B})^{(A\wedge B)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^u_A})^{A\downarrow}, (\boldsymbol{b}^{\boldsymbol{\tau}^u_B})^{B\downarrow} \\
(\boldsymbol{\phi}^{(\boldsymbol{\tau}^u_A \to \boldsymbol{\tau}^u_B)^*})^{(A\to B)\downarrow} & :\equiv & \lambda \boldsymbol{a}^{\boldsymbol{\tau}^d_A}.(\mathrm{U}^{\boldsymbol{\tau}^u_B}\{\boldsymbol{\phi}'(\boldsymbol{a}^{A\uparrow}) \ : \ \boldsymbol{\phi}' \in \boldsymbol{\phi}\})^{B\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^u_A}, c^{\sigma^*})^{(\exists z^{\mathrm{st}^\sigma} A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^u_A})^{A\downarrow}, c \\
(\boldsymbol{\phi}^{(\sigma^* \to \boldsymbol{\tau}^u_A)^*})^{(\forall z^{\mathrm{st}^\sigma} A)\downarrow} & :\equiv & \lambda c^{\sigma^*}.(\mathrm{U}^{\boldsymbol{\tau}^u_A}(\{(\boldsymbol{\phi}' c) \ : \ \boldsymbol{\phi}' \in \boldsymbol{\phi}\}))^{A\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\exists z^\sigma A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\downarrow} \\
(\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{(\forall z^\sigma A)\downarrow} & :\equiv & (\boldsymbol{a}^{\boldsymbol{\tau}^d_A})^{A\downarrow}
\end{array}
$$

P. Oliva, **Kreisel's modified realizability and recent variants**, to appear

Unifying Interpretation 2.0

$$
\begin{array}{lcl}
|P(x)|^a & \equiv & x \prec_P a \\
|A \otimes B|^{x,v}_{y,w} & \equiv & |A|^x_y \otimes |B|^v_w \\
|A \lozenge_z B|^{x,v}_{y,w} & \equiv & |A|^x_y \lozenge_z |B|^v_w \\
|A \multimap B|^{f,g}_{x,w} & \equiv & |A|^x_{g(x,w)} \multimap |B|^{f(x)}_w \\
|\forall z A(z)|^x_y & \equiv & \forall z |A(z)|^x_y \\
|\exists z A(z)|^x_y & \equiv & \exists z |A(z)|^x_y \\
|!A|^x_a & \equiv & \forall y \sqsubset a |A|^x_y
\end{array}
$$

one parameter for each atomic formula P

one parameter to deal with contraction

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\forall x^{\tau} A \equiv \forall x(\tau(x) \to A)$$
$$\exists x^{\tau} A \equiv \exists x(\tau(x) \wedge A)$$

Precise

$$|\tau(x)|^{a} \equiv x = a$$

Approximate

$$|\tau(x)|^{a} \equiv x \leq^{*} a$$

$$|\tau(x)|^{a} \equiv x \in a$$

...

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Unifying Interpretation 2.0

$$\forall x^{\tau,\mathrm{st}} A \quad \equiv \quad \forall x(\tau(x) \to \mathrm{st}(\mathrm{x}) \to A)$$
$$\exists x^{\tau,\mathrm{st}} A \quad \equiv \quad \exists x(\tau(x) \wedge \mathrm{st}(\mathrm{x}) \wedge A)$$

Type predicate / Standardness predicate

$$|\tau(x)| \quad \equiv \quad \tau(x)$$
$$|\mathrm{st}(x)|^a \quad \equiv \quad x \in a$$

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

$\forall y \sqsubset a \vert A\vert^x_y$	$x \prec_P a$	**Interpretation**
$\forall y \vert A\vert^x_y$	$\tau(x) \wedge (x = a)$	Kreisel modified realizability
$\forall y \in a \vert A\vert^x_y$	$\tau(x) \wedge (x = a)$	Diller-Nahm interpretation
$\vert A\vert^x_a$	$\tau(x) \wedge (x = a)$	Gödel's Dialectica interpretation
$\forall y \vert A\vert^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded modified realizability
$\forall y \leq^* a \vert A\vert^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded functional interpretation
$\forall y \vert A\vert^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand modified realizability
$\forall y \in a \vert A\vert^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand functional interpretation

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