Realizability

Lecture 2: Kreisel Modified Realizability

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Lecture 1: Homework

Find IL derivation of

$$\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$$

Homework

Find **IL** derivation of

$$(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)$$

Kleene Realizability

$$
\begin{aligned}
n \mathbf{\ r\ } s = t &\equiv s = t \\
n \mathbf{\ r\ } A \land B &\equiv (n_0 \mathbf{\ r\ } A) \land (n_1 \mathbf{\ r\ } B) \\
n \mathbf{\ r\ } A \lor B &\equiv (n_0 = 0 \to n_1 \mathbf{\ r\ } A) \land (n_0 \neq 0 \to n_1 \mathbf{\ r\ } B) \\
n \mathbf{\ r\ } A \to B &\equiv \forall m(m \mathbf{\ r\ } A \to \{n\}(m) \downarrow \land \{n\}(m) \mathbf{\ r\ } B) \\
n \mathbf{\ r\ } \forall x A &\equiv \forall m(\{n\}(m) \downarrow \land \{n\}(m) \mathbf{\ r\ } A[m/x]) \\
n \mathbf{\ r\ } \exists x A &\equiv n_1 \mathbf{\ r\ } A[n_0/x]
\end{aligned}
$$

Kleene, JSL, 1945

Realizability of Markov Principle

MP (Markov Principle).
$\forall m(\neg\forall k A_{\mathrm{qf}}(m,k) \to \exists k \neg A_{\mathrm{qf}}(m,k))$

Theorem. For any instance φ of **MP** there exists a numeral n s.t.

$$\mathbf{HA} + \mathbf{MP} \vdash n \ \mathbf{r} \ \varphi$$

Proof. Let n be such that $\{n\}(m) = \mu k \,.\, \neg A_{\mathrm{qf}}(m,k)$

Plan

- Lecture 1: Kleene (number) realizability
- **Lecture 2: Kreisel modified realizability**
- Lecture 3: General structure of realizability
- Lecture 4: Realizability of arithmetic and analysis

Lecture 2: Kreisel Modified Realizability

- Finite Types / Simply-type Lambda Calculus
- Heyting arithmetic all in finite types $\mathbf{HA}^{\omega}$
- Modified Realizability
- Soundness and Characterisation
- Applications

Finite Types

Finite Types

Def. The finite types $\mathcal{T}$ are inductively defined as

- $\mathbb{N} \in \mathcal{T}$ (*basic type*)
- If $\rho, \tau \in \mathcal{T}$ then $\rho \to \tau \in \mathcal{T}$ (*function types*)

Sometimes convenient to also assume:

- $\mathbb{B} \in \mathcal{T}$ (*booleans*)
- If $\rho, \tau \in \mathcal{T}$ then $\rho \times \tau \in \mathcal{T}$ (*product types*)

Typed λ-calculus

Gödel System T

Def. System **T** is an extension of the simply typed lambda calculus with new constants:

- $0 : \mathbb{N}$ (*zero*)
- $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ (*successor, write* $\mathrm{s}\,x$ *as* $x + 1$)
- $\mathrm{R}_\tau : \tau \to (\mathbb{N} \to \tau \to \tau) \to \mathbb{N} \to \tau$ (*Gödel primitive rec. of type* τ)

and axioms, e.g.

$$\mathrm{R}_\tau\, a\, f\, 0 = a$$

$$\mathrm{R}_\tau\, a\, f\, (n+1) = f\, n\, (\mathrm{R}_\tau\, a\, f\, n)$$

Heyting Arithmetic in all finite types $\mathbf{HA}^\omega$

Heyting Arithmetic in all finite types $\mathbf{HA}^\omega$

Terms: The terms of Gödel's system **T**

Atomic formulas: Equality between numbers: $s = t$

Axioms for equality relation and system **T** terms

Quantifiers for each finite type $\forall x^\tau A$ **and** $\exists x^\tau A$

Axiom schema for induction for extended language:

$$A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n) \quad \text{(IND)}$$

Modified Realizability

Modified Realizability

$$\begin{aligned}
\langle\rangle\ \mathbf{mr}\ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \wedge B &\equiv (\mathbf{x}\ \mathbf{mr}\ A) \wedge (\mathbf{y}\ \mathbf{mr}\ B) \\
n, \mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \vee B &\equiv (n = 0 \to \mathbf{x}\ \mathbf{mr}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ A \to B &\equiv \forall \mathbf{x}(\mathbf{x}\ \mathbf{mr}\ A \to \mathbf{fx}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ \forall z^{\tau} A &\equiv \forall z^{\tau}(\mathbf{f}z\ \mathbf{mr}\ A) \\
a, \mathbf{x}\ \mathbf{mr}\ \exists z^{\tau} A &\equiv \mathbf{x}\ \mathbf{mr}\ A[a/z]
\end{aligned}$$

Interpretations of AC and IP

AC (Axiom of Choice). $\forall x^\tau \exists y^\rho A(x, y) \to \exists f^{\tau \to \rho} \forall x^\tau A(x, fx)$

IP (Independence of Premise). $(\neg A \to \exists y^\rho B(y)) \to \exists y^\rho (\neg A \to B(y))$

Theorem. $\mathbf{HA}^\omega \vdash \mathbf{t}\ \mathbf{mr}\ \mathbf{AC}$, for a tuple of terms $\mathbf{t} \in \mathbf{T}$

Theorem. $\mathbf{HA}^\omega \vdash \mathbf{t}\ \mathbf{mr}\ \mathbf{IP}$, for a tuple of terms $\mathbf{t} \in \mathbf{T}$

Modified Realizability: Characterisation

AC (Axiom of Choice). $\forall x^\tau \exists y^\rho A(x, y) \to \exists f^{\tau\to\rho} \forall x^\tau A(x, fx)$

IP (Independence of Premise). $(\neg A \to \exists y^\rho B(y)) \to \exists y^\rho(\neg A \to B(y))$

Characterisation Theorem. $\mathbf{HA}^\omega + \mathbf{AC} + \mathbf{IP} \vdash A \leftrightarrow \exists \mathbf{x}(\mathbf{x} \text{ mr } A)$

$$
\begin{aligned}
\langle\rangle \ \mathbf{mr}\ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \wedge B &\equiv (\mathbf{x}\ \mathbf{mr}\ A) \wedge (\mathbf{y}\ \mathbf{mr}\ B) \\
n, \mathbf{x}, \mathbf{y}\ \mathbf{mr}\ A \vee B &\equiv (n = 0 \to \mathbf{x}\ \mathbf{mr}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ A \to B &\equiv \forall \mathbf{x}(\mathbf{x}\ \mathbf{mr}\ A \to \mathbf{f}\mathbf{x}\ \mathbf{mr}\ B) \\
\mathbf{f}\ \mathbf{mr}\ \forall z^\tau A &\equiv \forall z^\tau(\mathbf{f}z\ \mathbf{mr}\ A) \\
a, \mathbf{x}\ \mathbf{mr}\ \exists z^\tau A &\equiv \mathbf{x}\ \mathbf{mr}\ A[a/z]
\end{aligned}
$$

Soundness Theorem. If $\mathbf{HA}^\omega + \mathbf{AC} + \mathbf{IP} \vdash A$ then there are terms $\mathbf{t} \in \mathbf{T}$ such that $\mathbf{HA}^\omega \vdash \mathbf{t}\ \mathbf{mr}\ A$

Homework

MP (Markov Principle). $\forall m(\neg\forall n A_{\mathrm{qf}}(m,n) \to \exists n \neg A_{\mathrm{qf}}(m,n))$

Show, using **mr**, that

$$\mathbf{HA}^{\omega} \nvdash \mathbf{MP}$$

q-mr and mrt

Aczel Slash Translation

$$
\begin{aligned}
\Gamma \mid A_{\mathrm{at}} &\equiv \Gamma \vdash A_{\mathrm{at}} \\
\Gamma \mid A \wedge B &\equiv \Gamma \mid A \text{ and } \Gamma \mid B \\
\Gamma \mid A \vee B &\equiv \Gamma \mid A \text{ or } \Gamma \mid B \\
\Gamma \mid A \to B &\equiv (\text{if } \Gamma \mid A \text{ then } \Gamma \mid B) \text{ and } \Gamma \vdash A \to B \\
\Gamma \mid \forall x A &\equiv (\Gamma \mid A[n/z] \text{ for all numerals } n) \text{ and } \Gamma \vdash \forall x A \\
\Gamma \mid \exists x A &\equiv \Gamma \mid A[n/z], \text{ for some numeral } n
\end{aligned}
$$

Truth property. $\Gamma \mid A$ implies $\Gamma \vdash A$

Modified Realizability with Truth

$$\langle\rangle \ \mathbf{mrt}\ s = t \ \equiv\ s = t$$

$$\mathbf{x}, \mathbf{y}\ \mathbf{mrt}\ A \wedge B \ \equiv\ (\mathbf{x}\ \mathbf{mrt}\ A) \wedge (\mathbf{y}\ \mathbf{mrt}\ B)$$

$$n, \mathbf{x}, \mathbf{y}\ \mathbf{mrt}\ A \vee B \ \equiv\ (n = 0 \to \mathbf{x}\ \mathbf{mrt}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mrt}\ B)$$

$$\mathbf{f}\ \mathbf{mrt}\ A \to B \ \equiv\ \forall \mathbf{x}(\mathbf{x}\ \mathbf{mrt}\ A \to \mathbf{f}\mathbf{x}\ \mathbf{mrt}\ B) \wedge (A \to B)$$

$$\mathbf{f}\ \mathbf{mrt}\ \forall z^\tau A \ \equiv\ \forall z^\tau(\mathbf{f}z\ \mathbf{mrt}\ A) \wedge \forall z^\tau A$$

$$a, \mathbf{x}\ \mathbf{mrt}\ \exists z^\tau A \ \equiv\ \mathbf{x}\ \mathbf{mrt}\ A[a/z]$$

Truth property. $\mathrm{HA}^\omega \vdash (\vec{x}\ \mathbf{mrt}\ A) \to A$

Applications

Disjunction Property.
If $\mathbf{HA}^\omega \vdash A \vee B$ then either $\mathbf{HA}^\omega \vdash A$ or $\mathbf{HA}^\omega \vdash B$

Existence Property.
If $\mathbf{HA}^\omega \vdash \exists x^\tau A(x)$ then $\mathbf{HA}^\omega \vdash A(t)$, for some term t of system $\mathbf{T}$

Relative Consistency.
If $\mathbf{HA}^\omega$ is consistent then so is $\mathbf{HA}^\omega + \mathbf{AC} + \mathbf{IP}$

Elimination of AC/IP. Assuming $\mathbf{HA}^\omega \vdash A \leftrightarrow \exists \mathbf{x}(\mathbf{x}\ \mathbf{mr}\ A)$,
If $\mathbf{HA}^\omega + \mathbf{AC} + \mathbf{IP} \vdash A$ then $\mathbf{HA}^\omega \vdash A$