Realizability

Lecture 4: Arithmetic and Analysis

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Plan

- Lecture 1: Kleene (number) realizability
- Lecture 2: Kreisel modified realizability
- Lecture 3: General structure of realizability
- **Lecture 4: Realizability of arithmetic and analysis**

Lecture 4: Arithmetic and Analysis

- Classical logic and negative translations
- Friedman's A-translation
- Classical realizability
- Peano arithmetic
- Comprehension and classical analysis

Classical Logic

Intuitionistic Logic

$\neg A \equiv A \to \bot$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B}\ (\land\mathrm{I}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \lor A_2}\ (\lor\mathrm{E})$$

$$\frac{\Gamma \vdash A_1 \land A_2}{\Gamma \vdash A_i}\ (\land\mathrm{E}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\ (\to\mathrm{E}) \qquad \frac{\Gamma \vdash A \lor B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\lor\mathrm{E})$$

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{Ax}) \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ (\forall\mathrm{I}) \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\exists\mathrm{I})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\mathrm{EFQ}) \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t)}\ (\forall\mathrm{E}) \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(x) \vdash B}{\Gamma \vdash B}\ (\exists\mathrm{E})$$

Classical Logic = IL + DNE

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ (\mathrm{DNE})$$

Exercise: Complete this proof

Excluded middle

$$\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\neg(A \vee \neg A) \vdash \neg A} \quad \dfrac{\cdots}{\neg(A \vee \neg A) \vdash \neg\neg A}}{\neg(A \vee \neg A) \vdash \bot}\ (\to \mathrm{E})}{\vdash \neg\neg(A \vee \neg A)}\ (\to \mathrm{I})}{\vdash A \vee \neg A}\ (\mathrm{DNE})$$

Proof by contradiction

$$\dfrac{\dfrac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash \neg\neg A}\ (\to \mathrm{I})}{\Gamma \vdash A}\ (\mathrm{DNE})$$

Peano Arithmetic

Peano Arithmetic in all finite types PA^ω

Terms: The terms of Gödel's system T

Atomic formulas: Equality between numbers: $s = t$

Axioms for equality relation and system T terms

Quantifiers for each finite type $\forall x^\tau A$ **and** $\exists x^\tau A$

Axiom schema for induction for extended language:

$$A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n) \quad \text{(IND)}$$

Negative Translations

Negative Translations

$$P^N \equiv P$$
$$(A \wedge B)^N \equiv A^N \wedge B^N$$
$$(A \vee B)^N \equiv A^N \vee B^N$$
$$(A \to B)^N \equiv A^N \to B^N$$
$$(\forall x A)^N \equiv \forall x \neg\neg A^N$$
$$(\exists x A)^N \equiv \exists x A^N$$

Main Result.
$$\Gamma \vdash_{\mathbf{CL}} A \quad \Rightarrow \quad \Gamma^N \vdash_{\mathbf{IL}} \neg\neg A^N$$

Proof. Induction on the classical proof $\Gamma \vdash A$. Main task is to show that DNE is provable for A^N.

Friedman's A-translation

Friedman's A-Translation

$$
\begin{aligned}
\bot^A &\equiv A \\
P^A &\equiv P \vee A \\
(B \wedge C)^A &\equiv B^A \wedge C^A \\
(B \vee C)^A &\equiv B^A \vee C^A \\
(B \rightarrow C)^A &\equiv B^A \rightarrow C^A \\
(\forall x B)^A &\equiv \forall x B^A \\
(\exists x B)^A &\equiv \exists x B^A
\end{aligned}
$$

Main Lemma.
If $\Gamma \vdash_{\mathbf{IL}} B$ then $\Gamma^A \vdash_{\mathbf{IL}} B^A$

Π_2-Conservation Result.
$\vdash_{\mathbf{CL}} \forall x \exists y P(x, y) \Rightarrow \vdash_{\mathbf{IL}} \forall x \exists y P(x, y)$

Proof. Negative translation followed by A-translation for $A \equiv \exists y P(x, y)$

Classical Realizability

Modified Realizability

$$\begin{aligned}
\langle\rangle \ \mathbf{mr} \ \bot &\equiv \bot \\
\langle\rangle \ \mathbf{mr} \ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y} \ \mathbf{mr} \ A \wedge B &\equiv (\mathbf{x} \ \mathbf{mr} \ A) \wedge (\mathbf{y} \ \mathbf{mr} \ B) \\
n, \mathbf{x}, \mathbf{y} \ \mathbf{mr} \ A \vee B &\equiv (n = 0 \to \mathbf{x} \ \mathbf{mr} \ A) \wedge (n \neq 0 \to \mathbf{y} \ \mathbf{mr} \ B) \\
\mathbf{f} \ \mathbf{mr} \ A \to B &\equiv \forall \mathbf{x}(\mathbf{x} \ \mathbf{mr} \ A \to \mathbf{f}\mathbf{x} \ \mathbf{mr} \ B) \\
\mathbf{f} \ \mathbf{mr} \ \forall z^\tau A &\equiv \forall z^\tau(\mathbf{f}z \ \mathbf{mr} \ A) \\
a, \mathbf{x} \ \mathbf{mr} \ \exists z^\tau A &\equiv \mathbf{x} \ \mathbf{mr} \ A[a/z]
\end{aligned}$$

Classical Modified Realizability

Fix $\exists n P(n, \mathbf{m})$

$$
\begin{aligned}
n \ \mathbf{mr}\ \bot &\equiv P(n, \mathbf{m}) \\
\langle\rangle \ \mathbf{mr}\ s = t &\equiv s = t \\
\mathbf{x}, \mathbf{y} \ \mathbf{mr}\ A \wedge B &\equiv (\mathbf{x}\ \mathbf{mr}\ A) \wedge (\mathbf{y}\ \mathbf{mr}\ B) \\
n, \mathbf{x}, \mathbf{y} \ \mathbf{mr}\ A \vee B &\equiv (n = 0 \to \mathbf{x}\ \mathbf{mr}\ A) \wedge (n \neq 0 \to \mathbf{y}\ \mathbf{mr}\ B) \\
\mathbf{f} \ \mathbf{mr}\ A \to B &\equiv \forall \mathbf{x} (\mathbf{x}\ \mathbf{mr}\ A \to \mathbf{f}\mathbf{x}\ \mathbf{mr}\ B) \\
\mathbf{f} \ \mathbf{mr}\ \forall z^\tau A &\equiv \forall z^\tau (\mathbf{f}z\ \mathbf{mr}\ A) \\
a, \mathbf{x} \ \mathbf{mr}\ \exists z^\tau A &\equiv \mathbf{x}\ \mathbf{mr}\ A[a/z]
\end{aligned}
$$

Classical Modified Realizability

Main Result. For some term $t \in T$

$\vdash_{\mathbf{PA}^\omega} \forall \mathbf{m} \exists n P(n, \mathbf{m}) \Rightarrow \vdash_{\mathbf{HA}^\omega} \forall \mathbf{m} P(t\mathbf{m}, \mathbf{m})$

Proof. Assume

$\vdash_{\mathbf{PA}^\omega} \forall \mathbf{m} \exists n P(n, \mathbf{m})$

By negative translation

$\vdash_{\mathbf{HA}^\omega} \forall \mathbf{m}((\exists n P(n, \mathbf{m}) \to \bot) \to \bot)$

By the classical modified realizability we have a term t'

$\vdash_{\mathbf{HA}^\omega} \forall \mathbf{m}, f(\forall n(P(n, \mathbf{m}) \to P(fn, \mathbf{m})) \to P(t'f\mathbf{m}, \mathbf{m}))$

Classical Analysis

Analysis

Comprehension (CA). $\exists f^{\mathbb{N}\to\mathbb{N}} \forall n (fn = 0 \leftrightarrow A(n))$

Countable choice ($\mathbf{AC}_0$). $\forall n^{\mathbb{N}} \exists x^{\rho} A(n, x) \to \exists f^{\mathbb{N}\to\rho} \forall n A(n, fn)$

Lemma. $\mathbf{AC}_0 \vdash_{\mathbf{PA}^{\omega}} \mathbf{CA}$

Proof. Apply $\mathbf{AC}_0$ to **LEM**: $\forall n \exists k (k = 0 \leftrightarrow A)$

Double Negation Shift

Countable choice ($\mathbf{AC}_0$). $\forall n^{\mathbb{N}} \exists x^{\rho} A(n, x) \to \exists f^{\mathbb{N} \to \rho} \forall n A(n, fn)$

Double Negation Shift (DNS). $\forall n^{\mathbb{N}} \neg\neg A(n) \to \neg\neg \forall n A(n)$

Lemma. $\mathbf{AC}_0 + \mathbf{DNS} \vdash_{\mathbf{HA}^{\omega}} (\mathbf{AC}_0)^N$

Lemma. $\mathbf{PA}^{\omega} + \mathbf{CA}$ has a negative translation into $\mathbf{HA}^{\omega} + \mathbf{AC}_0 + \mathbf{DNS}$

Realizability of DNS

Double Negation Shift (DNS). $\forall k^{\mathbb{N}} \neg\neg A(k) \to \neg\neg \forall k A(k)$

Assume $\exists n P(n, \mathbf{m})$ fixed. We have efq $\mathbf{mr}\ \perp \to A(k)$

Given realisers

$$\phi_k\ \mathbf{mr}\ (A(k) \to \perp)) \to \perp)$$

$$q\ \mathbf{mr}\ (\forall k^{\mathbb{N}} A(k) \to \perp)) \to \perp)$$

find a realiser for $\perp$ (i.e. $\exists n P(n, \mathbf{m})$) via a *bar recursive* definition

$$\Phi(\phi, q)(s) =_{\mathbb{N}} q(\lambda k^{\mathbb{N}} . \mathrm{efq}(\phi_k(\lambda x^{A(k)} . \Phi(\phi, q)(s @ (k, x)))))$$